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Riemann Conjecture and Quantum Mechanics

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Abstract. Riemann extended $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, s = \sigma + it$, and took $s_0 = 1/2 + it$ to get real function $\xi(t) = G(s_0)\zeta(s_0)$, $G(s) = (s-1)\pi^{-s/2}\Gamma(\frac{s}{2}+1)$, and consider t as complex $z = t - i\beta$, $\beta = \sigma - 1/2$. He proposed Riemann Conjecture (RC): All zeros of $\xi(z)$ are real (i.e. $\beta = 0$). Using all real roots $\{t_n\}$ and complex roots $\{z_j\}$ of $\xi(z)$, we have a unique product $\xi(t) = w(t)Q(t)$ on $z = t$, $w(t) = \xi(0)\prod_{n=1}^{\infty}(1 - t^2/t_n^2)$. We prove $|\zeta(s_0)| = \frac{|w(t)|}{M(t)} \leq 3t^{1/4}$, $t \geq T_1 = 10^3$, $M(t) = |G(s_0)|$. If $\xi(z)$ has complex roots z_j with $R_1 = |z_1| > T_1$. Taking $h = 4\pi$, it is easy to prove $Q(t) \leq (4h^2+1)R_1^{-2}$ in $J_h = [R_1 - h, R_1]$. Finally, there is at least one $t_0 \in J_h$ such that $1 \leq |\zeta(1/2 + it_0)| = \frac{|w(t_0)|}{M(t_0)}Q(t_0) \leq 3(4h^2+1)T_1^{1/4-2} \leq 0.0107$. As calculations confirm no complex roots for $t \leq T_1$, this contradiction proves RC. Nextly, the relation between RC and quantum mechanics is discussed. The generalized RC and some suggestions are provided.

Keywords Riemann conjecture, Zeta-function, Xi-function, product expression, quantum mechanics

AMS Classification of Subjects 11M26, 65E05, 81Q05

§1 Introduction

Riemann (1859) proposed the Riemann Conjecture(RC), which has remained unsolved for 160 years. In recent 50 years, one found that RC is closely related to quantum mechanics. So Riemann conjecture becomes a common interest of mathematicians and physicists. Why is it so difficult? We find that there are two different research routes.

D. Hilbert (1900) proposed to study the Euler series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots, \quad s = \sigma + it, \quad \sigma > 1, \tag{1.1}$$

and the current classical formulation

Riemann hypothesis(RH). *All non-trivial zeros of $\zeta(s)$ have real part $\sigma = 1/2$.*

In 2000, Clay Mathematics Institute published Millennium 7 Problems, including RH, see the comments [1, 2, 3]. In the 20th century, mathematicians completed over ten large-scale calculations that confirmed the validity of RH to the height $t = 10^{12}$, see [4, 5, 6, 7]. They have used two algorithms, using Riemann-Siegel formula calculates real roots, using Euler expansion searches for complex roots. These numerical experiments have enhanced our confidence, but all efforts of proving RH have failed.

The first question is how to analytically extend (1.1). J. Conrey [3] pointed out that *"It is my belief that RH is a genuinely arithmetic question that likely will not succumb to methods of analysis."* He listed 15 mainstream analytic extensions of $\zeta(s)$, none of which were explicitly expressed. However, $\zeta(s)$ is not entire function, studying the series $\zeta(s) \neq 0$ is beyond the existing analysis ability, because the addition structure lacks a good contradiction mechanism. Actually we should consider entire function $\xi(z)$ and multiplication structure.

This paper will follow Riemann's route [8](pp.300-302) to study $\xi(z)$ and RC. Riemann extended $\zeta(s)$ and took $s_0 = 1/2 + it$ to get a real function

$$\xi(t) = G(s_0)\zeta(s_0),\ G(s) = (s-1)\pi^{-s/2}\Gamma(\frac{s}{2}+1),\ \ M(t) = |G(s_0)| = C_0 t^{7/4} e^{-\pi t/4}, \tag{1.2}$$

which is symmetric and conjugate. He considered t as complex $z = t - i\beta, \beta = \sigma - 1/2$(i.e. transform $s = 1/2 + iz$), then $\xi(z)$ is an entire function of order 1. He proposed

Riemann conjecture(RC). *All zeros t_j of $\xi(z)$ are real(i.e. $\beta = 0$).*

We briefly describe the idea of proof. We first use product expression to prove that RC is equivalent to the order: $|\xi(t - i\beta)| > |\xi(t - i\beta_0)|, if\ |\beta| > |\beta_0|$(Theorem 1). But the product has a prerequisite: all roots must be given in advance. By using all the real root $\{t_n\}$ and complex root $\{z_j\}$, then $\xi(z)$ on the real axis $z = t$ has a unique product expression

$$\xi(t) = w(t)Q(t), \quad w(t) = \xi(0)\prod_{n=1}^{\infty}(1 - \frac{t^2}{t_n^2}),\ \ Q(t) = \prod_{j=1}^{\infty} q_j(t), \tag{1.3}$$

These two factors, $w(t)$ and $Q(t)$, are absolutely convergent, independent of each other, and can be determined separately.

The proof of RC can be completed in the following three steps on the real axis $z = t$.

S1). Assuming that $\xi(z)$ has only real roots, then $\xi(t) = w(t), Q(t) = 1$, using the known upper bound $|\zeta(s_0)| \le 3t^{1/4}$(Lemma 1), we obtain **a basic estimate(Lemma 5)**

$$|w(t)|/M(t) = |\zeta(s_0)| \le 3t^{1/4}, \quad t \ge 100, \tag{1.4}$$

which is independent of complex roots, and valid in complex root case, see Remark 2 in §4.

S2). Assuming that $\xi(z)$ has complex roots z_j with the first modulus $R_1 = |z_1| > T_1 = 10^3$. Taking $h = 4\pi$, it is easy to prove that the factor

$$Q(t) \le q_1(t) \le (4h^2 + 1)R_1^{-2} << 1, \quad t \in J_h = [R_1 - h, R_1], \tag{1.5}$$

decays quadratically(Lemma 3). This is the havoc of complex roots.

S3). To use contradiction, we need a **selection principle**: there is at least one point $t_0 \in J_h$ such that $|\zeta(1/2+it_0)| \ge 1$. Finally, by combining the above three estimates, we get a contradiction

$$1 \le |\zeta(1/2 + it_0)| = \frac{|w(t_0)|}{M(t_0)} Q(t_0) \le 3(4h^2 + 1)T_1^{1/4-2} < 0.0107, \quad t_0 \in J_h. \tag{1.6}$$

Because the calculation has confirmed no complex roots in $t \le T_1 = 10^8$, therefore RC holds.

We believe that this contradictory method can be applied to more general problems (generalized RC), and suggest a feasible method for physicists.

In the second part of this paper, we discuss the connection between RC and quantum mechanics. And discuss several related issues. Finally, introduce some progression of finite element study for Hamilton systems in §8.

§2 Follow Riemann's research route

§2.1 Euler function $\zeta(s)$

Riemann took the Euler product of primes as a starting point,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in primes} (1 - \frac{1}{p^s})^{-1}, \quad s = \sigma + it, \ \sigma > 1. \tag{2.1}$$

and had

Theorem A. *The Euler function $\zeta(s)$ is analytic and has no roots when $\sigma > 1$.*

Proof. This only proves the latter. It is difficult to study the series $\sum n^{-s} \ne 0$ directly, so we consider the product on the right. Using $1 - p^{-\sigma} \le |1 - p^{-s}| \le 1 + p^{-\sigma}$, we have

$$|\zeta(s)| = \prod_p |1 - p^{-s}|^{-1} \ge \prod_p \frac{1}{1 + p^{-\sigma}} \ge \prod_{n=1}^{\infty} \frac{1}{1 + n^{-\sigma}} > 0.$$

This proof is so concise by multiplication structure. We find that this idea of using product is beneficial for studying RC. Besides, Riemann proved that the analytic continuation of $\zeta(s)$ has some symmetry about $\sigma = 1/2$, so the study of RC can be limited in a critical domain $0 \le \sigma \le 1$.

§2.2 Riemann function $\xi(z)$

Riemann took $s = \sigma + it, \sigma > 1$ and $y = n^2\pi x$, the gamma function has

$$\Gamma(\frac{s}{2}) = \int_0^\infty y^{s/2-1}e^{-y}dy = n^s\pi^{s/2}\int_0^\infty x^{s/2-1}e^{-n^2\pi x}dx,$$

Sum up to obtain (this is the third extension, see §8.1)

$$\zeta(s) = \sum_{n=1}^\infty n^{-s} = \pi^{s/2}\Gamma^{-1}(\frac{s}{2})\int_0^\infty x^{s/2-1}\psi(x)dx, \quad \psi(x) = \sum_{n=1}^\infty e^{-n^2\pi x}. \tag{2.2}$$

Using Jacobi's identity $2\psi(x) + 1 = x^{-1/2}(2\psi(\frac{1}{x}) + 1)$, Riemann got an integral representation

$$\zeta(s) = \pi^{s/2}\Gamma^{-1}(\frac{s}{2})\{\frac{1}{s(s-1)} + \int_1^\infty (x^{s/2-1} + x^{-(s+1)/2})\psi(x)dx\}, \tag{2.3}$$

which is analytically extended over the whole plane except a pole $s = 1$.

Multiplying (2.3) by $G(s)$, Riemann directly took $s = 1/2 + it$ and defined

$$\xi(t) = G(s)\zeta(s), \quad G(s) = (s-1)\pi^{-s/2}\Gamma(\frac{s}{2} + 1), \ |G(s)| > 0. \tag{2.4}$$

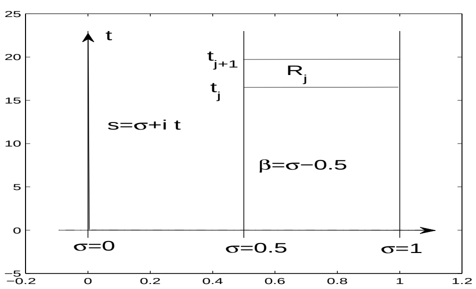
Inserting ζ into (2.4) and applying integration by parts twice, he had [8] p.17,

$$\begin{aligned}\xi(t) &= \frac{1}{2} + \frac{s(s-1)}{2}\int_1^\infty (x^{s/2-1} + x^{-s/2-1/2})\psi(x)dx,\\ &= r_1 + \int_1^\infty (x^{s/2-1} + x^{-s/2-1/2})(2x^2\psi'' + 3x\psi')dx,\end{aligned}$$

where $r_1 = \frac{1}{2} + \psi(1) + 4\psi'(1) = 0$. Then Riemann got an even real function[8]pp.301-302,

$$\xi(t) = 4\int_1^\infty \cos(\frac{t}{2}\ln x)(x^{3/2}\psi'(x))'x^{-1/4}dx, \tag{2.5}$$

He considered t as complex variable. In fact, Riemann used translation $\beta = \sigma - 1/2$ and rotation $s = 1/2 + iz, \ z = t - i\beta$, see Figure 1, and got an entire function $\xi(z)$ by (2.5).



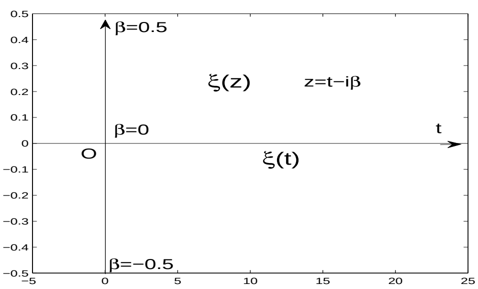


Figure 1. Translation $\beta = \sigma - 1/2$ and rotation $z = t - i\beta$

Remark 1. Using $s = 1/2 + it$ is a clever idea to simplify symmetry, which clearly distinguishes between real roots and complex roots. We know that complicated mathematical formulas can often be transformed into a standard form using coordinate transformations, which not only deepen our understanding of the essence of mathematics, but also help discover new properties. Some scholars have accepted another notation $\xi(s) = G(s)\zeta(s)$, [8] p.17, but Riemann's notation $\xi(t)$ is more concise and inspiring in research, see (2.5),(2.6),(2.7) and (2.8).

Theorem B. *The entire function $\xi(z) = G(s)\zeta(s)$ has an integral expression*

$$\xi(z) = \int_1^\infty \cos(\frac{z}{2}\ln x)x^{-3/4}\sum_{n=1}^\infty (4n^2\pi^2x^2 - 6n^2\pi x)e^{-n^2\pi x}dx, \ z = t - i\beta. \tag{2.6}$$

which is symmetric $\xi(z) = \xi(-z)$ and conjugate $\xi(\overline{z}) = \overline{\xi(z)}$.

Riemann proved that the number of roots in symmetric domain $\Omega_T = \{|\beta| \le 1/2, 0 \le |t| \le T\}$ is about $N = \frac{T}{2}(\ln\frac{T}{2\pi} - 1)$ and proposed RC.

Riemann conjecture(RC). *All the roots of function $\xi(z)$ are real.*

§2.3 Product expression $\zeta(z)$

Finally Riemann proved

Theorem C. *If $\{z_j\}$ are all roots of $\xi(z)$, there is a product expression*

$$\xi(z) = \xi(0)\prod_{j=1}^\infty(1 - \frac{z^2}{z_j^2}), \quad z = t - i\beta, \quad \xi(0) = 0.497120778.... \tag{2.7}$$

Riemann didn't explain whether z_j is real or complex. This just is a key of the whole problem.

§2.4 Equivalence of RC

Inspired by A.Hinkkanen[9](1997) and J.Lagarias[10](1999), we have proved [12](2022)

Theorem 1. *Riemann conjecture(RC) is equivalent to the order of $|\xi(z)|$ with respect to β,*

$$|\xi(t - i\beta)| > |\xi(t - i\beta_0)|, \quad if \quad |\beta| > |\beta_0| \ge 0. \tag{2.8}$$

Proof. Obviously, (2.8) leads to RC. Conversely, if RC holds, using product expression $\xi(z) = \xi(0)\prod_{j=1}^\infty(1 - z^2/t_j^2), \ z = t - i\beta$, and taking logarithm and derivative, we have

$$\frac{D_\beta\xi(z)}{\xi(z)} = -i\frac{\xi'(z)}{\xi(z)} = -i\sum_{j=1}^\infty\{\frac{t - t_j + i\beta}{(t - t_j)^2 + \beta^2} + \frac{t + t_j + i\beta}{(t_j + t)^2 + \beta^2}\},$$

and its real part

$$J = Re(\frac{D_\beta\xi(z)}{\xi(z)}) = \beta\sum_{j=1}^\infty\{\frac{1}{(t_j - t)^2 + \beta^2} + \frac{1}{(t_j + t)^2 + \beta^2}\} > 0, \quad \beta > 0.$$

Denoting $\xi(z) = u + iv$ and $D_\beta \xi = u_\beta + iv_\beta$, we rewrite

$$J = Re\frac{D_\beta\xi(z)\bar{\xi}(z)}{|\xi(z)|^2} = \frac{\psi(t,\beta)}{|\xi(z)|^2} > 0, \quad \psi(t,\beta) = uu_\beta + vv_\beta > 0.$$

To cancel the derivative, using $|\xi|^2 = u^2 + v^2$ and $D_\beta|\xi|^2 = 2\psi(t,\beta)$, we have

$$|\xi(t - i\beta)|^2 - |\xi(t - i\beta_0)|^2 = 2\int_{\beta_0}^{\beta} \psi(t,\beta)d\beta > 0, \ for\ \beta > \beta_0 \geq 0.$$

Similarly discuss $\beta < 0$ and $\psi(t,\beta) < 0$. Therefore Theorem is proved.

This equivalence theorem makes us believe that using the contradiction of multiplication structure is the key to proving RC. Ancient Greek philosopher Aristotle thought that *"Order and symmetry are important elements of beauty"*. Now we can say that *"Symmetry and order are mathematical beauty of Riemann Conjecture"*.

§3 Riemann-Siegel formula and selection principle

§3.1 Riemann-Siegel formula is valid for the complex z

Using asymptotic expansion of $\Gamma(s)$,

$$\Gamma(s) = \sqrt{2\pi}s^{s-1/2}e^{-s}(1 + \frac{1}{12s} + \frac{1}{288s^2} - \frac{139}{51740s^3} + O(s^{-4})), \quad |\arg(s)| < \pi. \tag{3.1}$$

For $s = \sigma + it = 1/2 + iz$ we have

$$\begin{cases} G(s) &= \frac{s(s-1)}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) = M(z)e^{i\phi(z)}, \qquad \xi(z) = G(s)\zeta(s), \\ M(z) &= (\frac{\pi}{2})^{1/4}t^{7/4}e^{-t\pi/4}(\frac{t}{2\pi})^{-\beta/2}(1 + O(t^{-2})) \\ \phi(z) &= \frac{t}{2}(\ln\frac{t}{2\pi} - 1) + \frac{7\pi}{8} + \frac{1}{48t} + \frac{\beta\pi}{4} - \frac{7\beta + \beta^2}{4t} + O(t^{-2}). \end{cases} \tag{3.2}$$

where $M(z)$ decays exponentially. Define Riemann formula[8]p.119-120,

$$Z(z) = e^{i\phi(z)}\zeta(s), \quad s = 1/2 + iz, \quad z = t - i\beta. \tag{3.3}$$

If $\beta = 0$, the $e^{i\phi(t)}$ is a symmetrizer, as $Z(t)$ is a real function, and then $|Z(t)| = |\zeta(1/2 + it)|$.

C.Siegel[11](1932) discovered an important formula in Riemann's manuscript unpublised.

Theorem D(Riemann-Siegel formula). *For all* $t \in [10, \infty)$, *there is*

$$\begin{cases} Z(t) &= 2\sum_{n=1}^{N}\frac{\cos(\phi_1(t) - t\ln n)}{n^{1/2}} + R(t), \quad 2\pi N^2 \leq t < 2\pi(N+1)^2, \\ \phi(t) &= \frac{t}{2}(\ln\frac{t}{2\pi} - 1) + \frac{7\pi}{8} + \frac{1}{48t} + \frac{7}{5760t^3} + ... \\ R_N(t) &= (-1)^N(\frac{t}{2\pi})^{-1/4}\{C_0 + C_1t^{-1/2} + C_2t^{-1} + C_3t^{-3/2} + C_4t^{-2} + ...\} \end{cases} \tag{3.4}$$

where $N = [X]$ is the integer part of $X = (t/2\pi)^{1/2}$ and the coefficients

$$\begin{cases} C_0 & = \psi(q) = \dfrac{\cos(q^2 + 3\pi/8)}{\cos(\sqrt{2\pi}q)}, \quad q = (\sqrt{t} - N\sqrt{2\pi}) - \sqrt{\pi/2}, \quad |q| \leq \sqrt{\pi/2}, \\ C_1 & = -\dfrac{1}{24}\psi'''(q), \quad C_2 = \dfrac{1}{2^4}(\psi''(q) + \dfrac{1}{72}\psi^{(6)}(q)), ..., \end{cases} \tag{3.5}$$

Here $\psi(q)$ is an analytic function with a removable singularity $q_0 = \pm\sqrt{\pi/2}/2$. Our method transforms the singularity q_0 to the origin, $x = q - q_0$, $\phi(q) = \sin(x^2 + 0.5\sqrt{2\pi}x)/\sin(\sqrt{2\pi}x)$. In a small interval $x \in [-\delta, \delta]$, using Taylor expansion reduces the factor x. We know that $0.3827 = \sin\pi/8 \leq \psi(q) \leq \cos(\pi/8) = 0.9239$. $|\psi'(q)| \leq 1$, $|\psi''(q)| \leq 1, |\psi'''(q)| \leq 2$. The remainder $R_N(t)$ is an asymptotic series, for $t \geq 100$, just calculate the first item. Now $Z(t)$ is the most effective tool for calculating real roots, as long as it takes $N = [(t/2\pi)^{1/2}]$.

Riemann calculated the first several real roots, e.g., $t_1 = 14.135, t_2 = 21.022, t_3 = 25.011$. It seems that Riemann adopted the method of combining theoretical analysis (i.e. symmetry and conjugation) with computations.

Recall that the function $\xi(t)$ is valid for complex z. We wake up to the following

Theorem 2. *R-S formula $Z(z)$ is valid for the complex z.*

This result is obvious for Riemann, but very important for us. Hereafter, the R-S complex formula provides a unified and efficient algorithm. We indeed derived a complex formula $Z_1(t, \beta)$ using Riemann's method (It needs 15 pages, omitted here), and found that the $Z_1(t, \beta)$ is same as the $Z(z)$, while the derivation of real formula $Z(t)$ is more concise. We have seen once again what important the Riemann's idea of simplifying symmetry is.

By (3.2) and (3.3), we have $M(z) = (\frac{t}{2\pi})^{-\beta/2}M(t)(1 + O(t^{-2}))$ and

$$Z(z) = e^{i\phi(z)}\zeta(s) = \frac{\xi(z)}{M(z)} = (\frac{t}{2\pi})^{\beta/2}\frac{\xi(z)}{M(t)}(1 + O(t^{-2})). \tag{3.6}$$

As $|\xi(z)|/M(t)$ has the order and $(\frac{t}{2\pi})^{\beta/2}$ increases in β, so $|Z(z)|$ also has the order

$$|Z(t - i\beta)| > |Z(t - i\beta_0)|, \quad if \quad |\beta| > |\beta_0| \geq 0. \tag{3.7}$$

We will return to this extremely important property in §6.5.

The function $Z(t)$ is high-frequency oscillation and unbounded[8] p.184.

Lemma 1(Upper bound)[12]. *$\zeta(s)$ is unbounded, and $|\zeta(1/2 + it)| \leq 3t^{1/4}$ for $t \geq 100$.*

Proof. Edwards [8]p.200, proved $Z(t) = O(t^{1/4})$, which is obvious for Riemann. $R(t)$ is an asymptotic series. For $t \geq 100$, it is sufficient to calculate the first two terms, we get a better bound

$$\begin{aligned} |R(t)| & \leq (\frac{t}{2\pi})^{-1/4}\{0.9239 + \frac{1}{120} + \frac{1}{1600}(1 + \frac{58}{72}) + ...\} \leq (\frac{t}{2\pi})^{-1/4} < 1, \\ |Z(t)| & \leq 2(1 + \sum_{n=2}^{N} n^{-1/2}) + |R(t)| \leq 2(1 + \int_1^N x^{-1/2}dx) + 1 \\ & < 2(1 + 2(N^{1/2} - 1)) + 2 = 4N^{1/2} \leq 4(\frac{t}{2\pi})^{1/4} < 2.5265t^{1/4}. \end{aligned}$$

The Lemma is proved.

Remark 2. In analytic number theory, there is a better estimate $|\zeta(1/2+it)| \le Ct^{1/6}$. However, these estimates are conservative, and in fact the growth of $Z(t)$ is very slow. We guess $|Z(t)| \le |\ln t|^{\mu}, \mu \le 3$, using data of $t \le 1.13e+6$ [6](with $\mu = 1.5$) and figures of high peaks at $t = 1.6e+8$ [5](with $\mu = 1.65$).

§3.2 Selection principle

Through data analysis and theoretical research, we have proved that the maximum spacing of the highest peak $|Z(p)|$ is $D \le 1.5\pi$. We call $P_d = 4\pi$ quasi-period. In a quasi-period, the high and low peaks will re-appear in different forms. $Z(t)$ is an irregular oscillation, the height $|Z| = 1$ is a basic level. To use contradictions, we also need a reliable positive lower bound.

Theorem 3(Selection principle). *For $T \ge T_1 = 10^3$ and $h = 4\pi$, there is at least one point $t_0 \in J_h = [T-h, T]$ such that $|Z(t_0)| \ge 1$.*

Proof. Rewrite R-S formula as $Z(t) = 2\cos\phi(t) + u(t)$. We take $T = 10^3$ and consider $\phi'(T) = \frac{1}{2}\ln\frac{T}{2\pi}$ as a constant, then $f(t) = 2\cos\phi(t)$ is a periodic oscillation with spacing(semi-periodicity)$d(T) = 2\pi/\ln\frac{T}{2\pi} = 1.2393$. The length $h = 4\pi$ includes $m = h/d(T) = 10.14$ spacings. For simplicity, we take $m = 10$ even. Thus, the point set of $f(t) \ge 1$ or $f(t) \le -1$ is J^+ or J^-, with measure $h/3$, respectively. On the other hand, as $u(t)$ is random oscillation, the point of $u(t) \ge 0$ falling in J^+ has probability of about $1/6$, and the point of $u(t) \le 0$ falling in J^- has probability of about $1/6$, so the probability measure of $|Z(t)| \ge 1$ in J_h is about $h/3$. Therefore there is at least one point $t_0 \in J_h$ such that $|Z(t_0)| \ge 1$. The theorem is proved.

In order to better understand the properties of $|Z(t)| \ge 1$, we calculate the height $|Z(p_j)|$ at $p_j = (t_{j+1}+t_j)/2$ using the first $N = 2e+6$ zeros t_j [6]. Divide N into $k = 20$ groups, with $1e+5$ zeros in each group. The percentage of $|Z(p_j)| \ge 1$ for each group is more stable, all exceeding 63%. These computations strongly support the principle of selection.

§4 A Simplified Proof of the Riemann conjecture

We know the following two results:

1. Using Theorem A and symmetry, all roots of $\xi(z)$ can be limited in $\Omega_\infty = \{0 \le t < \infty, |\beta| \le 1/2\}$. This is a premise for RC true.

2. Calculation has confirmed that $\xi(z)$ has no complex roots in $\Omega_T = \{0 \le t \le T, |\beta| \le 1/2\}$, $T = 10^3$. This is necessary for the contradiction.

Thus all roots are limited in $\Omega_\infty \backslash \Omega_T$ and the whole proof can be finished on $z = t$. This is a significant simplification.

§4.1 Compression effect of complex roots

If $\xi(z)$ has a complex root $z_j = t'_j + i\alpha_j, 0 < |\alpha_j| \leq 1/2$, and its modulus $R_j = |z_j| > T_1 = 10^3$ (Now no complex roots have been found to the height $t = 10^{12}$,[5, 7]). Due to the symmetry and conjugation, it has four conjugate complex roots $z_j = \pm(t'_j \pm i\alpha_j)$ and 4th degree factor

$$q_j(z) \quad = (1 - \frac{z^2}{(t'_j + i\alpha_j)^2})(1 - \frac{z^2}{(t'_j - i\alpha_j)^2}) = (1 - \frac{z^2}{R_j^2})^2 + \frac{4z^2\alpha_j^2}{R_j^4},\ z = t - i\beta,$$

which is a positive real function on $z = t$

$$q_j(t) = (1 - \frac{t^2}{R_j^2})^2 + (\frac{t}{R_j})^2\mu_j,\ \mu_j = (\frac{2\alpha_j}{R_j})^2 < 1. \tag{4.1}$$

Lemma 2. $q_j(t) < 1$ *for* $t \in (0, R_1]$.

Proof. As $q_j(0) = 1$ and $q_j(R_j) = \mu_j \leq R_j^{-2} << 1$. consider derivative in $I_1 = (0, R_j]$

$$q'_j(t) = -4\frac{t}{R_j^2}(1 - \frac{t^2}{R_j^2}) + 2\frac{t}{R_j^2}\mu_j = -2\frac{t}{R_j^2}\{2(1 - \frac{t^2}{R_j^2}) - \mu_j\} = 0,$$

and $q_j(t)$ attains its minimum at a point $t_0^2/R_j^2 = 1 - \frac{1}{2}\mu_j$, see figure 2.

$$q_j(t_0) = (1 - \frac{t_0^2}{R_j^2})^2 + (\frac{t_0}{R_j})^2\mu_j = \frac{1}{4}\mu_j^2 + (1 - \frac{1}{2}\mu_j)\mu_j = \mu_j\{1 - \frac{1}{4}\mu_j\} < \mu_j.$$

So the Lemma is proved.

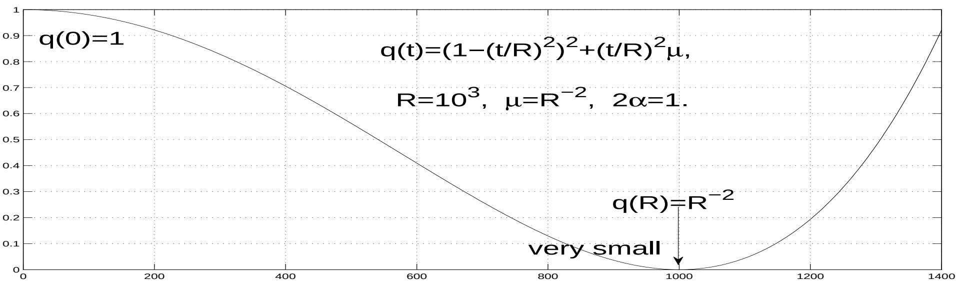


Figure 2. Taking $R = 20$, the curve $q_1(t) < 1$ in $(0, R]$.

Lemma 3. *For* $R_1 > T_1 = 10^3$ *and* $h = 4\pi$, *we have*

$$q_1(t) \leq (4h^2 + 1)R_1^{-2} << 1, \quad t \in J_h = [R_1 - h, R_1]. \tag{4.2}$$

Proof. For $t \in J_h$, directly calculate the polynomial

$$q_1(t) = (1 - \frac{t^2}{R_1^2})^2 + \frac{t^2}{R_1^2}\mu_j \leq (1 - (1 - h/R_1)^2)^2 + R_1^{-2} \leq (4h^2 + 1)T_1^{-2} \leq 6.33e - 4.$$

So $q_1(t)$ has quadratic decay in J_h, this is the havoc of complex roots. The Lemma is proved.

§4.2 Proof of Riemann conjecture

Assume that $\xi(z)$ has finite or infinite many complex roots $\{z_j\}$. Arrange $T_1 < R_1 \leq R_2 \leq ... \leq R_n \leq \cdots$. So using all real roots $\{t_n\}$ and complex roots $\{z_j\}$, based on Theorem C, we have a basic expression

$$\xi(t) = w(t)Q(t), \quad w(t) = \xi(0)\prod_{n=1}^{\infty}(1-(\frac{t}{t_n})^2), \quad Q(t) = \prod_{j=1}^{\infty} q_j(t) > 0, \tag{4.3}$$

which are absolutely convergent, guaranteed by Riemann zeros $z_n = O(n \ln^{-1} n)$ [8] p.42. The advantage of this multiplication structure is that $w(t)$ and $Q(t)$ are independent of each other and can be studied separately.

Lemma 4(Basic lemma). *The product $w(t)$ constructed by all real roots has an estimate*

$$\frac{|w(t)|}{M(t)} \leq 3t^{1/4}, \qquad t \geq 100. \tag{4.4}$$

and which is valid in the case of complex roots.

Proof . As $\xi(t) = w(t)Q(t)$, $w(t)$ and $Q(t)$ can be determined separately. If $\xi(z)$ has only real roots, then $Q(t) = 1$ and $\xi(t) = w(t)$. Using Lemma 1 in §3, we get $|Z(t)| = |w(t)|/M(t) \leq 3t^{1/4}$. The second conclusion can be proved by uniqueness of product expression. Assuming that $\xi(t) = w(t)Q(t)$ has complex roots, $Q(t) \neq 1$, but the factor $w(t)$ remains unchanged, the estimate (4.4) is still valid. Also see Remark 3. The Lemma is proved.

Using Lemma 2, all $q_j(t) \leq 1$ in $[0, R_1]$, and using Lemma 3, we have

$$Q(t) \leq q_1(t) \leq (4h^2+1)R_1^{-2} << 1, \quad t \in J_h = [R_1 - h, R_1]. \tag{4.5}$$

Using (4.3), (4.4) and (4.5), we get an important estimate

$$|Z(t)| = \frac{|\xi(t)|}{M(t)} = \frac{|w(t)|}{M(t)}Q(t) \leq 3t^{1/4}q_1(t) \leq 3(4h^2+1)R_1^{1/4-2}, \quad t \in J_h. \tag{4.6}$$

To use contradiction, we need the selection principle(Theorem 3). For $R_1 > T_1 = 10^3$ and $h = 4\pi$, there is at least one point $t_0 \in J_h = [R_1 - h, R_1]$ such that

$$1 \leq |Z(t_0)| \leq 3(4h^2+1)R_1^{1/4-2}, \quad t_0 \in J_h. \tag{4.7}$$

where R_1 is unknown, but R_1 has negative power $R_1^{1/4-2}$ on right side. If R_1 is replaced by $T_1 = 10^3$, it is amplified, and we obtain a contradiction

$$1 \leq 3(4h^2+1)T_1^{\lambda-2} < 0.0107 << 1, \quad t_0 \in J_h. \tag{4.8}$$

Because calculations confirmed that $\xi(z)$ has no complex roots in $t \leq T_1$, so RC holds.

Remark 3. We directly proved $|w(t)|/M(t) \leq Ct^{\lambda}$ with $\lambda = 1.0678$ using all real roots $\{t_n\}$, but requiring 18 pages, it is difficult for physicists to read. For this, this paper employs the basic

Lemma. Some scholars argue that obtaining (4.4) assumes no complex roots, why is it valid for complex roots? Is this a logical error? We explain as follows. People have long been accustomed to studying contradictions in additive structures. Although using series $\xi(z)$ obtains $w(t)$ and (4.4), but cannot exclude complex roots. It is impossible to prove RC by studying $\xi(z) \neq 0$(The same dilemma arises in studying $\zeta(s) \neq 0$). But the product $\xi(t) = w(t)Q(t)$ clearly distinguishes between real roots and complex roots, here $w(t)$ is (4.4) and unique. We combine the advantages of both in order to prove the basic Lemma. This logic is new and reasonable. The breakthrough of scientific problems requires not only innovative methods, but also new logical thinking. We recall that when Riemann extended $\zeta(s)$, it is necessary to assume $\sigma > 1$, but after obtaining the integral (2.3), it is found to be effective in the whole plane. People have never doubted its premise $\sigma > 1$. We leave this logical question for readers to comment on.

§5 Calculation can detect the clues of proving RC

§5.1 Calculation is the third thinking

In the 4th and 5th centuries BC, two different styles of mathematical systems emerged independently from the East and the West. The ancient Greek "Elements" established a mathematical system of logical reasoning, while Chinese ancient mathematics classic "Nine Chapters on Mathematics" (about 6th century BC)[15] emphasized the construction algorithms and solved nine classes of practical problems, unlike Greek mathematics. They each developed independently for thousands of years, and the 3rd century AD was a turning point in history. Greek mathematics was besieged and destroyed, and withdrew from the historical stage. After the Renaissance in the 14th century, it was rediscovered and developed into modern mathematics. And in China, the greatest mathematician LiuHui (225-295 AD) emerged, who made over a hundred annotations (263 AD) for the "Nine Chapters", reaching the summit of Chinese mathematical thought (which continued to develop until the 15th century and later declined). We find that Liuhui made five innovations: The limit conception (calculated the area $\pi_{192} = 3.14$, rather than the perimeter of Archimedes), extrapolation(using infinite series summation, obtain $E_x\pi_{192} = (4\pi_{192} - \pi_{96})/3 = 3.1416$). Richardson(1927) proposed extrapolation), matrix elimination(Gauss proposed elimination), element method(Archimedes had the germ of elemental method) and average slope(This is a distinctive concept in ancient China), which are close to calculus.

Next, we discuss mathematical methodology. R.Descartes pointed out that *"There are two ways of thinking: intuition and logic"*. H.Poincare emphasized that *" Logic is a tool for proof. Intuition is the tool of invention"*. But faced with so difficult RH, these two ideas are ineffective. We have to turn to Liuhui, who pointed out in his preface of “Nine Chapters” (263 AD):

"Analyze reasoning with logic, understand the essence with figure"(i.e. logic and intuition).

"Calculation can recognize subtle and detect the unknown".

LiuHui's methodology has a profound influence on me. When studying the Riemann conjecture, many insights and inspirations often come from computational intuition, which is just the third thinking needed in computer era.

Selberg pointed out that, *"There have probably been very few attempts at proving the Riemann conjecture, because, simply, no one had ever had any really good idea for how to go about it"*. How do we get started? The only way is to follow LiuHui's idea, *"computing can detect the unknown"*.

§5.2 General Euler expansion

For further research purposes, we first derive the Euler expansion in a concise manner.

Assuming the real function $f(x)$ and its derivatives decay rapidly, denoting by $F(x)$ its primitive function,

$$\int_N^\infty f(x)dx = F(x)|_N^\infty = -F(N). \tag{5.1}$$

For example, $f(x) = x^{-s}, f(x,a) = (x+a)^{-s}$ and so on. Consider infinite series

$$J = \sum_{n=1}^\infty f(n) = \sum_{n=1}^{N-1} f(n) + J_N, \quad J_N = \sum_{n=N}^\infty f(n). \tag{5.2}$$

After adding and subtracting $-F(N)$, using integration by parts, then

$$\begin{aligned} J_N &= \sum_{n=N}^\infty f(n) = \sum_{n=N}^\infty \int_0^1 (f(n) - f(n+x))d(x - \frac{1}{2}) - F(N) \\ &= -F(N) + \frac{1}{2}f(N) - \sum_{n=N}^\infty \int_0^1 f'(n+x)(x - \frac{1}{2})dx, \end{aligned}$$

where separating main part $\int_0^1 xdx = 1/2$ such that $\int_0^1 (x - \frac{1}{2})dx = 0$(orthogonal to 1). Its integral $g_2(x) = \int_0^x (x - \frac{1}{2})dx = \frac{1}{2}(x^2 - x)$ has $g_2(0) = g_2(1) = 0$. Using integration by parts, we have

$$J_N \quad = -F(N) + \frac{1}{2}f(N) + 0 + \sum_{n=N}^\infty \int_0^1 f''(n+x)g_2(x)dx.$$

Continuing with the three methods, namely, the separating the main part, the integration by parts, and the remaining orthogonal polynomials (in fact, this is a gradual process of orthogonalization), Euler obtained the famous Euler expansion

$$\begin{cases} J &= \sum_{n=1}^{N-1} f(n) + S_{2m} + R_N, \\ S_{2m} &= -F(N) + \frac{1}{2}f(N) + \frac{B_2}{2!}f'(N) + \frac{B_4}{4!}f'''(N) + \cdots + \frac{B_{2m}}{(2m)!}f^{(2m-1)}(N), \\ R_{N,2m} &= \frac{1}{(2m)!}\sum_{n=1}^\infty \int_0^1 f^{(2m)}(N+x)B_{2m}(x)dx, \end{cases} \tag{5.3}$$

where B_j is the Bernoulli number,

$$\begin{aligned} B_2 &= \frac{1}{6}, \ B_4 = -\frac{1}{30}, \ B_6 = \frac{1}{42}, \ B_8 = -\frac{1}{30}, \ B_{10} = \frac{5}{66}, \ B_{12} = -\frac{691}{2730}, \\ B_{14} &= \frac{7}{6}, \ B_{16} = -\frac{3617}{510}, \ B_{18} = \frac{43867}{798}, \ B_{20} = -\frac{174611}{330}, \cdots. \end{aligned} \tag{5.4}$$

and $B_j(x)$ is Bernoulli polynomial. The remainder $R_{N,2m}$ is very complicated, we have

$$|R_{N,2m}| \le \frac{|B_{2m}|}{(2m)!}\int_N^\infty |f^{(2m)}(x)|dx \approx \frac{|B_{2m}|}{(2m)!}|f^{(2m-1)}(N)|. \tag{5.5}$$

When take $N \ge t+2m$ and $2m \ge 16$, the remainder $R_{N,2m} \sim 10^{-12}$ can be ignored.

Remark 3. The formula (5.3) is often extended over $Re(s) > 2-2m$. Here we also encountered a similar logical problem. Although the integral (5.1) may diverge, but people have abandoned $F(\infty)$ in (5.3) (see also (5.6)), so its analytic extension is defined by (5.3). This logic is widely accepted in analytic extensions. We specifically explain it.

§5.3 Euler expansion $\zeta(s)$ and its estimates

The Euler function $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ yields $2m$-order formula

$$\begin{cases} \zeta(s) &= \zeta_N(s) + S_{2m} + r_{2m}, \quad \zeta_N(s) = \sum_{n=1}^{N-1} n^{-s}, \quad s = \sigma + it, \\ S_{2m} &= \{\frac{N}{s-1} + \frac{1}{2} + \frac{B_2}{2}sN^{-1} + \cdots + \frac{B_{2m}}{(2m)!}s(s+1)...(s+2m-2)N^{-2m+1}\}N^{-s}, \\ r_{2m} &= -\frac{s(s+1)\cdots(s+2m-1)}{(2m)!}\sum_{j=N}^\infty \int_0^1 B_{2m}(x)(j+x)^{-s-2m}dx, \end{cases} \tag{5.6}$$

This formula has been extended to $Re(s) > -2m+2$. The remaining term r_{2m} is very complicated, and there is a concise estimate

$$|r_{2m}| \le K_{2m}(\frac{|s|+2m-1}{N})^{2m}N^{1-\sigma}, \quad K_{2m} = \frac{|B_{2m}|}{(2m)!(2m+\sigma-1)}.$$

It is easy to calculate the value of the constant K_{2m},

K_{14}=4.59e-11	K_{16}=9.91e-13	K_{18}=2.19e-14	K_{20}=4.912e-16	K_{22}=1.12e-17

So we have the following

Calculation criteria. *If take $2m = 14 \sim 22$ and $N \ge t+2m$, we can abandon the remainder r_{2m} and still have high accuracy.*

§5.4 Why cannot RH be proved using $\zeta = U + iV$?

Figure 3 shows curves U(solid line) and V(dashed line) for $\sigma = 0.5, 0.6, 0.8, 1.0$. We see that

1). $\{U, V\}$ are the irregular oscillation in $\sigma = 1/2$, without symmetry. And U and V are almost tangent at many points. This is the reason why RH cannot be proved near $\sigma = 1/2$.

2). The whole curve $U(t, 0.6)$ has shifted upwards by a small distance such that number of its zeros suddenly decrease. The curve V still has many zeros, but (U, V) no longer has a common zero. Therefore, studying infinite series $\zeta(s) \ne 0$ for $1/2 < \sigma \le 0.6$ is almost impossible.

3). The curve $U(t, \sigma)$ for $\sigma \ge 0.8$ has deviated from t-axis, although V still has many zeros. It is not surprising to obtain some new results using this property $|U(t, \sigma)| > c_0 > 0$.

Based on the above calculations and analysis, we believed(2017)that

Conclusion 1. *Using $\zeta(s)$ to prove RH is impossible and have to give up.*

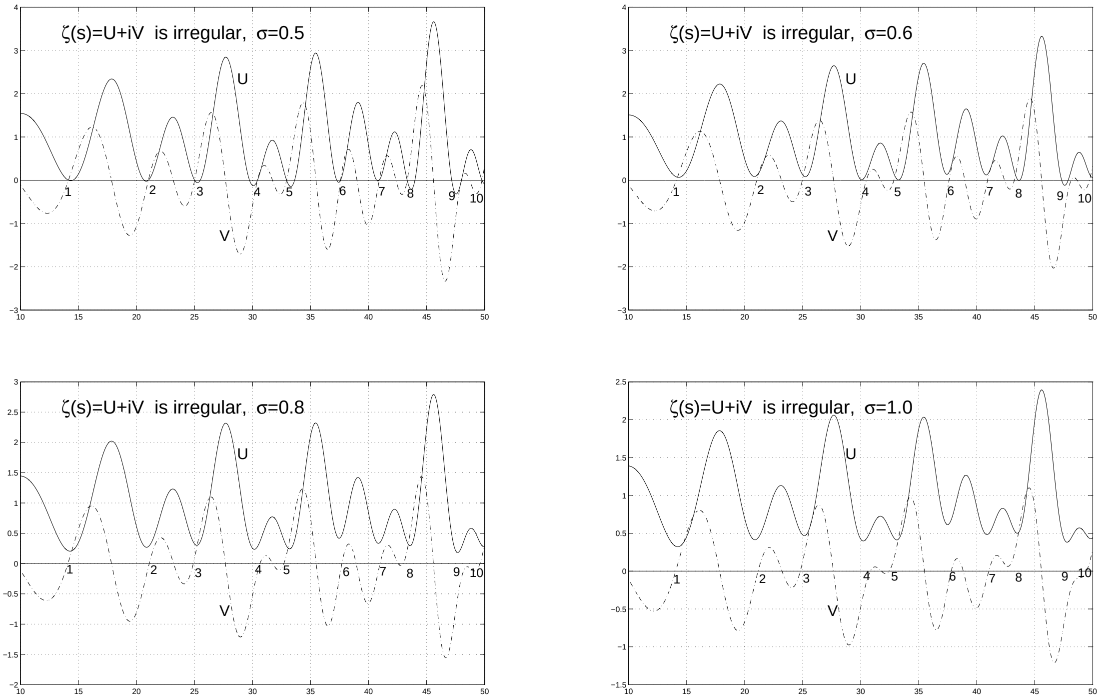


Figure 3. $\{U, V\}$ is not alternative oscillation for $\sigma \in [0.5, 1.0]$.

§5.5 $Z(z) = u + iv$ is an alternating oscillation

There are three methods to calculate $Z(z)$.

1. We once directly studies and calculated the Riemann integral $\xi(z)$. Due to the exponential decay of $G(s)$, $\xi(z)$ is not suitable for large-scale calculations.

2. We found that R-S formula $Z(z)$ is effective for $z = t - i\beta$, which is a unified and efficient algorithm. As this algorithm has not been generalized to general case and will not be used here.

3. Use Euelr expansion to calculate $\zeta(s)$, and then symmetrize $Z(z) = e^{i\phi(z)}\zeta(s) = u + iv$. We would rather use this algorithm for future needs.

Figure 4 depicts the change of (u, v). We have seen that

1). For $\beta = 0$, the imaginary part $v(t, 0) \equiv 0$, the real part $u(t, 0)$ is high-frequency oscillation and there are the first 10 zeros.

2). For $\beta = 0.1, 0.3, 0.5$, the curve $\{u(t, \beta), v(t, \beta)\}$ is alternating oscillation, geometrically implying that RC holds.

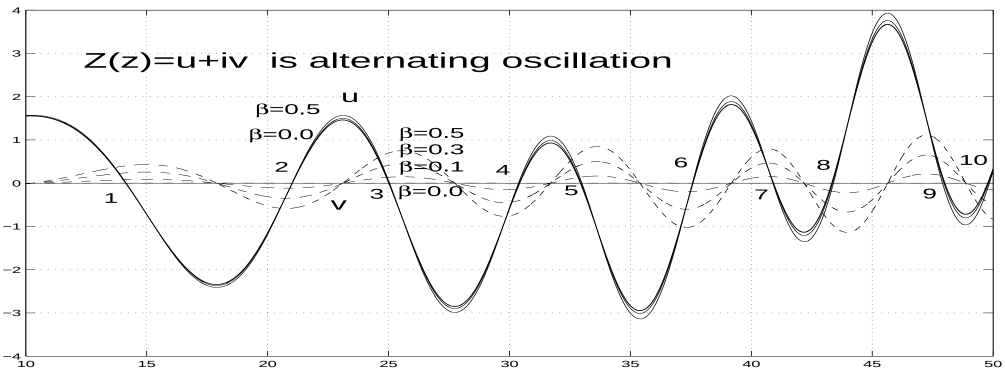


Figure 4. $\{u(t,\beta), v(t,\beta)\}$ alternatively oscillates for $\beta > 0$.

§5.6 The order of $|Z(z)|$ is equivalent to RC

According to (3.6), $|Z(z)| = (\frac{t}{2\pi})^{\beta}|\zeta(s)|$ also has the order. Figure 5 shows the order, confirming the correctness of RC. The biggest advantage of calculating order is that for each t, only four values $\beta = 0, 0.1, 0.3, 0.5$ need to be calculated. This is the most economical method.

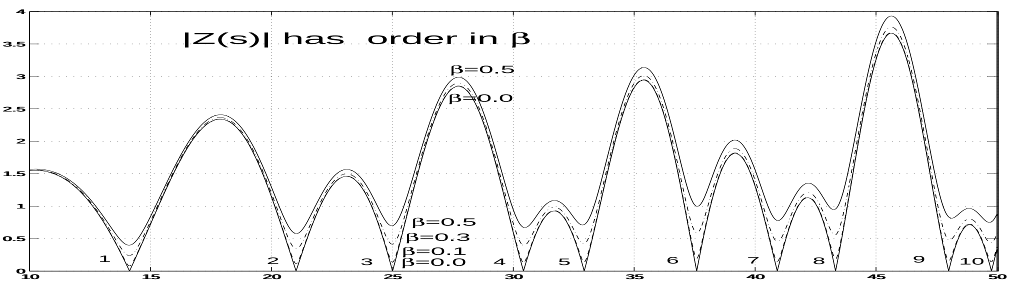


Figure 5. The order of $|Z(t - i\beta)|$ for $\beta = 0, 0.1, 0.3, 0.5$.

We once used analytical methods such as saddle point method and asymptotic expansion to study RC, but both have failed. Why? Because in the integral expression (2.6), for any complex $t = z$, there is always $\cos(\frac{z}{2}\ln x) = 1$ at $x = 1$. By using integration by parts many times, it can be proved that $\xi(z)$ decays to any negative power, but it cannot be proved that $\xi(z)$ decays exponentially, let alone RC. After three years of research, we have realized that,

Conclusion 2. *RC cannot be proved alone using Riemann functions ξ.*

In 2022, we proved that RC is equivalent to the order of $\xi(z)$(Theorem 1), and discovered the compressibility of complex root product, see (4.2). Finally, we recognized that

Conclusion 3. *The structural contradiction of product expression of complex roots is the key to proving RC.*

How to provide a rigorous analysis proof, it took us 4 years to find the correct method.

§6 Generalized RC and a feasible method

§6.1 Generalization of RC

What factors are essential in the proof of RC? Can it be applied to a wider range of issues? (still use $s = 1/2 + iz$, where $z = t - i\beta, \beta = \sigma - 1/2$). We propose

Generalized RC(gRC). *Assume that the following conditions are satisfied*

C1). Generating function $g(s)$ is analytic and has no zeros if $\sigma > 1$ (likely with a pole $s = 1$).

C2). By Euler expansion of $g(s)$, calculation confirms that $g(s)$ has no complex roots in $\Omega_T = \{|t| \le T, |\beta| \le 1/2\}$, e.g. $T = 10^3$. This is a prerequisite for studying gRC.

C3). The analytic extension

$$f(z) = G(s)g(s), \quad z = t - i\beta, \tag{6.1}$$

is a symmetric and conjugate entire function.

C4). On real axis, $|f(t)|/|G(1/2 + it)| \le Ct^\lambda, \lambda < 2$ for $t \ge T$, e.g. $T = 10^3$.

Then all zeros of $f(z)$ are real, and $|f(z)|$ has the order

$$|f(t - i\beta)| > |f(t - i\beta_0)| \ge 0, \quad if \quad |\beta| > |\beta_0| \ge 0. \tag{6.2}$$

So the generation function $g(s)$ determines everything. We explain these conditions as follows.

1. How to ensure that condition C1 is the first key. Many scholars of number theory discuss the case where $g(s)$ has Euler product similar to (2.1). There are several examples to illustrate that if $g(s)$ does not have Euler product, condition C1 may not hold.

2. The condition C2 must be verified through calculation. If an anomaly is found in a finite interval, GRC is no longer valid.

3. Proving condition C3 is quite difficult. A feasible method for testing the order will be discussed in §6.5. Of course, once it is proved that $f(z)$ is an entire function, then product expression, structural contradiction, and equivalence theorem, are all its corollaries.

4. The condition C4 is needed for contradiction. The selection principle can be satisfied, if taking $|f(t_0)| \ge c_0$ for some smaller $c_0 > 0$. These conditions can also be determined by calculating $g(s)$. When starting to study GRC, we don't need to get involved in these troubles.

§6.2 Grand RH

Sarnak [2] mentioned Dirichlet series $L(s, \chi)$. Let χ be a primitive Dirichlet character of modulus (i.e. $\chi(mn) = \chi(m)\chi(n), \chi(1) = 1, \chi(n + bq) = \chi(n)$), the L be defined by Euler product

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}, \tag{6.3}$$

As with zeta, $L(s, \chi)$ extends to an entire function and satisfies the functional equation

$$\Lambda(s, \chi) := \pi^{-(s+a_\chi)/2}\Gamma(s/2 + a_\chi)L(s, \chi) = \epsilon_\chi q^{1/2-s}\Lambda(1 - s, \bar{\chi}). \tag{6.4}$$

where $a_\chi = (1+\chi(-1))/2$ and ϵ_χ is the sign of a Gauss sum. There is **Grand RH:** *All zeros of* $\Lambda(s,\chi)$ *have real part* $s = 1/2$. It is related to Goldbach conjecture. We believe GRH holds, because the most crucial conditions C1 and C3 have been satisfied, but C2 and C4 still need to be confirmed. He also added an interesting footnote, *"Hardy(Collected Papers, Vol.1,p.560) assures us that latter will be proven within a week of a proof of the RH"*. Of course, it may not be so simple, and it needs a strict proof.

§6.3 A counterexample

To explain the importance of the condition C1, we cite a counterexample in [16](pp.210-213). Using Hurwitz zeta-function $\zeta(s,a) = \sum_{n=1}^{\infty}(n+a)^{-s}$, $0 < a < 1$, and its linear combination with $\tan\theta_g = \frac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1} = 0.284079...$,

$$g(s) = \frac{1}{5^s}\{\zeta(s,\frac{1}{5}) - \zeta(s,\frac{4}{5}) + \tan\theta_g(\zeta(s,\frac{2}{5}) - \zeta(s,\frac{3}{5}))\},$$

(Note that it includes the difference of two $\zeta(s,a_j)$, then $s=1$ is not a pole, see Euler expansion (3.3)) They got an entire function

$$f(s) = (\frac{\pi}{5})^{-s/2}\Gamma(\frac{s+1}{2})g(s).$$

and proved three results as follows:

1). The number of roots on critical line $\{\sigma = 1/2, t \le T\}$ is $T\exp(c(\ln\ln\ln\ln T)^{1/2})$.

2). There are infinitely many roots in $\{1/2 < \sigma < 1, 0 < t < \infty\}$.

3). Outside the critical band $\{\sigma > 1, t \le T\}$, the number of roots exceeds cT, where $c > 0$ is a positive number [17].

How delicate this example is! They claimed that this is *"a counterexample"* and *"it does not satisfy RH"*. But we think this is incorrect, because they overlooked an important fact that the generating function $g(s)$ itself has complex roots in $Re(s) > 1$, which has already disrupted RH (This property is mentioned in item 13 in [16] p.240, and proved in [17]). To confirm this fact, we use Euler expansion of $g(s)$ and find the first complex root $s = 1.2019 + 22.639i$.

§6.4 The importance of Euler product

Example 1. $\zeta_1(s) = \zeta(s) - 1$ has no Euler product. Using Euler expansion, we find the first complex root $s = 1.407788 + i23.32799$ outside critical band, and the first complex root $s = 0.594417 + i42.39947$ inside critical band. So gRC does not hold.

Example 2. $\zeta(s,a) = \sum_{n=1}^{\infty}(n+a)^{-s}, 0 < a < 1$ has no Euler product. Taking $f(x) = (x+a)^{-s}$, its primitive function $\int(x+a)^{-s}dx = \frac{1}{1-s}(x+a)^{1-s}$. The N in S_{2m} should be $N+a$. We take $a = 0.5$ and find the first complex root $s = 1.29479 + i128.8194$ outside critical band, and the first complex root $s = 0.97359 + i18.8265$ inside critical band. So gRC does not hold.

§6.5 A feasible method

To discuss gRC, the first requirement is that the series $g(s)$ is extended to the whole plane by Euler expansion and calculation confirms no zeros in $|z| \leq T_1$ except for $\sigma = 1/2$.

The second difficulty is how to obtain the analytic extension $f(z)$? For many important problems, even mathematicians have not yet found their entire function expression. Can we continue our research without the expression? Sure. We shall suggest a feasible method as follows.

The order is the most crucial property. In (3.6), using $\xi(z) = M(z)e^{i\phi(z)}\zeta(s)$ can define $Z(z) = e^{i\phi(z)}\zeta(s)$, its modulus $|Z(z)| = (\frac{t}{2\pi})^{\beta/2}|\zeta(s)|$ still has the order, so RC holds, see figure 4. If we replace $\beta/2$ with β, then $|Z(z)| = (\frac{t}{2\pi})^{\beta}|\zeta(s)|$ still has the order. Thus the thinking can be applied to more general entire functions.

In §6.2, $L(s,\chi)$ is extended to the entire function $\Lambda(s,\chi)$, using $\Gamma(s/2+a)$ and $\pi^{-(s+a_\chi)/2}$. We find that similar gamma functions appear in many studies. We don't care about their modulus $M(t)$, but only focus on its argument, in a general form

$$\psi(t) = Kt(\ln\frac{t}{2\pi} - 1) + c, \quad \psi'(t) = K\frac{t}{2\pi}, \quad K \geq 1/2. \tag{6.5}$$

which has super-linear growth, and K is a selectable parameter. We write $f(z) = M(z)e^{i\psi(z)}g(s)$ in gRC and expand the argument $\psi(z) \approx \psi(t) - i\beta\psi'(t)$, then we can construct a new function

$$F(z) = e^{i\psi(z)}g(s), \quad |F(z)| = (\frac{t}{2\pi})^{K\beta}|g(s)|, \quad z = t - i\beta. \tag{6.6}$$

If the calculation confirms that $|F(z)|$ has the order, then gRC holds. For this, we only need to calculate $\beta = 0, 0.1, 0.2, 0.3, 0.5$ for each t. This is the feasible method for further research, no matter what specific form the entire function $f(z)$ takes.

Why do we emphasize the order so much? Because the order describes the minimization of modulus $|F(t-i\beta)|$, this is extremely important in both mathematics and physics. We recall that Riemann studied the distribution of prime numbers, introducing a counting function $Li(x) = \int_1^x \frac{dx}{\ln x}$, and taking the Riemann zero $\rho_j = 1/2 + i\alpha_j$ in the series $\sum(Li(x^{\rho_j}) + Li(x^{1-\rho_j})$, actually its modulus is minimized.

Physicists aim to discover new phenomena and properties, therefore computational detection and geometrical intuition are the most important weapons, should not be constrained by logical reasoning. We recall that Archimedes was a greatest sage nurtured by Euclidean geometry, but he ignored Euclid's logical reasoning and freely used various tools to do what he wanted.

§7 Relation between Riemann conjecture and quantum mechanics

Famous physicist F.Dyson [18](2009) reported on the Einstein lecture "Birds and Frogs" at AMS, which expounds scientific ideas. We only select the part related to RH. He said(p.214),

"It is likely that any proof of the Riemann hypothesis will likewise lead to a deeper understanding of many diverse areas of mathematics and perhaps of physics too. Riemann's zeta function,

and other zeta-functions similar to it, appears ubiquitously in number theory, in the theory of dynamical systems, in geometry, in the function theory, and in physics. The zeta-function stands at a junction where paths lead in many directions. A proof of the Riemann hypothesis will illuminate all the connections."

Now RC is no longer mysterious. As pointed out in §4, to prove RC, we found that the contradiction of multiplication structure of ξ is the most powerful tool, rather than the addition structure of ζ. This is completely beyond people's expectations. However, we often encounter ζ and its analogues in quantum mechanics. It seems that physical research(using additive structure) and mathematical proof(using multiplication structure) live in two different worlds. We strive to combine them and provide some suggestions for physicists as reference.

§7.1 Statistical quantum mechanics

The function $\zeta(s)$ has three types of extensions. Taking $x = ny$, gamma function

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx = n^s\int_0^\infty y^{s-1}e^{-ny}dy, \quad Re(s) > 1, \tag{7.1}$$

and $(1-r)^{-1} = 1 + r + r^2 + \cdots$ for $|r| < 1$, Riemann directly obtained the first expression

$$\zeta(s) = \sum_{n=1}^\infty n^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty y^{s-1}\sum_{n=1}^\infty e^{-ny}dy = \frac{1}{\Gamma(s)}\int_0^\infty \frac{y^{s-1}}{e^y - 1}dy, \ Re(s) > 1. \tag{7.2}$$

And use its contour integral to extend over the whole plane, except for one pole $s = 1$.

Nextly, for $Re(s) > 1$, rewrite

$$\zeta(s) \ = \sum_{n=1}^\infty n^{-s} = 1 - 2^{-s} + 3^{-s} - 4^{-s} + \cdots + 2^{1-s}\zeta(s).$$

Using (7.1) can get the second expression

$$\zeta(s) = \frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x + 1}dx, \quad Re(s) > 0. \tag{7.3}$$

It is difficult to large-scale calculate (7.2) and (7.3) because $\Gamma(s)$ decays exponentially $e^{-t\pi/2}$.

Based on the study of the distribution relationship between black body radiation energy density $E\nu$ and frequency ν, Plank(1900) proposed experimental model

$$E_\nu = \frac{c_1\nu^3}{e^{c_2\nu/T} - 1},$$

here T is the absolute temperature, c_1 and c_2 are constants. This formula is simple and consistent with experimental data (e.g. temperature up to $6000K$, wavelength $100 \sim 3000nm$). He proposed the 'quantum' hypothesis: For radiation of a certain frequency, an object can only absorb or emit at integer multiples of the minimum energy unit ϵ (a quantum). The energy of a quantum is $\epsilon = \hbar\nu$, where the Plank constant $\hbar = 6.6261 \times 10^{-34}$ joule $\cdot$ second.

In statistical quantum mechanics, the distribution function of multi-particle system is

$$\begin{cases} f(\epsilon) &= \dfrac{1}{e^{\frac{\epsilon-\nu(\theta)}{\theta}}-1}, \quad \text{Particles with integer spin(bosons)} \\ f(\epsilon) &= \dfrac{1}{e^{\frac{\epsilon-\nu(\theta)}{\theta}}+1}, \quad \text{Particle with half integer spin(fermion)} \end{cases} \tag{7.4}$$

here ϵ is the energy of a single particle, $\theta = k_B T$ is the thermodynamic equilibrium energy per unit energy (where k_B is the Bolzmann constant), $\nu(\theta)$ is chemical energy, and $f(\epsilon)$ is the probability density of energy particles found between ϵ and $\epsilon + d\epsilon$. Assuming the system does not obey the external field, it is $0 < \epsilon < \epsilon_{max}$. If $f(\epsilon)d\epsilon$ is the number of states of a single energy particle, then the total number of particles is

$$N(\theta) = \int_0^\infty \frac{f(\epsilon)}{e^{\frac{\epsilon-\nu(\theta)}{\theta}} \pm 1} d\epsilon, \quad \text{if } f(\epsilon) = 0 \text{ for } \epsilon > \epsilon_{max}, \tag{7.5}$$

and internal energy is

$$E(\theta) = \int_0^\infty \frac{\epsilon f(\epsilon)}{e^{\frac{\epsilon-\nu(\theta)}{\theta}} \pm 1} d\epsilon. \tag{7.6}$$

In the special case of $\nu = 0$, using coordinate transformation $x = \epsilon/\theta$, the above integrals can be represented by the $\zeta(s)$ function (7.2) or (7.3). Thus they are closely linked to RH. In the (+) case of $\nu \neq 0$, the asymptotic expansion can be used.

June 2023, Italian physicist M.Colozoo has discussed ζ with me many times. I am very grateful for his introduction to statistical quantum mechanics and its expressions.

§7.2 Stochastic quantum mechanical system

In quantum mechanics, the state of microscopic particle system can be described by a probability wave function $\psi(t, x, y, z)$. Schrödinger (1926) derived the quantum mechanics equations

$$i\hbar D_t \psi = \hat{H}\psi, \quad \hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(x, y, z),$$

here $\hat{H}$ is Hamiltonian operator. As it can be represented by a random Hermite matrix, the energy levels of the ensemble are determined by their eigenvalues. Therefore, their solution can be written formally as

$$\psi(t) \equiv \psi(t, x, y, z) = \exp(-\frac{\hat{H}t}{i\bar{h}})\psi_0, \quad \psi_0 = \psi(0, x, y, z).$$

From viewpoint of mathematical physics, using eigenpairs $\{\lambda_j, \phi_j\}$, any solution can be written as

$$\psi(t) = \sum_{j=1}^{\infty} e^{i\lambda_j t}(\phi_j, \psi_0)\phi_j(x, y, z).$$

The zeros of function $\psi(t)$ and the eigenvalue λ_j are two different concepts.

There was a rumor in mathematics group that in the 1920s, Hilbert-Polya associated the zeros of the ζ function with the eigenvalues of a Hermite matrix. But there is no literature record, and it has even been forgotten. Perhaps due to the unresolved issue of RH, it was given renewed attention after 1972. On December 8, 1981, A.Odlyzko wrote a letter to Polya inquiring about this matter. Fortunately, Polya was still alive at the age of 94 (he unfortunately passed away three years later), and replied confirming that this incident occurred in 1914, see [20]p.95. It is now known as

Hilbert-Polya conjecture. *All non-trivial zeros of $\zeta(s)$ correspond to the eigenvalues of a random Hermite matrix.*

Nowadays, HPC is a popular problem in physics and mathematics. But I don't understand why we need to associate the eigenvalues with the zeros of $\zeta(s)$?

Physical problems in the real world, such as quantum mechanics, have their own characteristics, and some of them have new structures and properties that are truly valuable. If we reverse the formulation of the problem, like the HPC, and require its eigenvalues to conform to the Riemann zeros, does this research have practical significance? It's likely another misleading!

§7.3 Eigenvalues of quasi-crystalline system

D. Shechtman (1984) discovered quasi-crystals, which are a special type of new material between crystals and non crystals (without translational periodicity, but with positional order), which have different properties in thermal conductivity, electrical conductivity and optics, and have been widely applied. F.Dyson [18](2009,p215) provided an authoritative description of quasi-crystals. For example, *"Here comes the connection of the one dimensional quasi-crystals with the Riemann hypothesis. If the Riemann hypothesis is true, then the zeros of zeta-function form one dimensional quasi-crystal according to the definition. They constitute a distribution of point masses on a straight line, and their Fourier transform is likewise a distribution of point masses, one at each of the logarithms of ordinary prime numbers and prime-power numbers....[19]"*. We don't understand why the point distribution of one-dimensional quasi-crystals must be arranged according to Riemann zeros. We don't think it is necessary to cater to Riemann's zeros.

Besides, Dyson said: *"Suppose that we find one of the quasi-crystals in our enumaration with the properties that identify it with the zeros of the Riemann zeta-function. Then we have proved the Riemann hypothesis ..."*. We think this may be a misunderstanding. Because the core of the Riemann hypothesis is to prove that ζ does not have non-trivial zeros $\sigma \neq 1/2$. So far, no counterexample has been found to the height $t = 10^{12}$. However calculations and experiments always are finite, but not mathematical proof of RH. We feel that physicists do not need to be involved in such 'theoretical proof'.

§8 Continuous finite element study for Hamilton systems

V.Arnold[21] divided the classical mechanics into three periods: Newton mechanics, Lagrange variation and Hamilton systems. Quantum mechanics coincides with Hamiltonian systems, characterized by an antisymmetric structure, i.e., a symplectic structure. In Euclid-geometry, introduce column vectors X, the identity matrix I and bi-linear form

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X^T IY = x_1 y_1 + x_2 y_2, \quad X^T IX = |X|^2.$$

Thus $\sqrt{X^T IX} = |X|$ is the length. In the symplectic structure, define the column vector X, the symplectic matrix J and the symplectic bi-linear form

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X^T JY = x_1 y_2 - x_2 y_1, \quad X^T JX = 0,$$

If a matrix M satisfies $M^T JM = J$(or $MJM^T = J$), called a symplectic matrix. The symplectic conservation corresponds to the conservation of angular momentum in Newton mechanics.

§8.1 Ordinary differential Hamilton system

We consider the ordinary differential Hamiltonian systems

$$p_t = -H_q(p,q), \; q_t = H_p(p,q), \quad p(0) = p_0, \; q(0) = q_0,$$

its solution $z(t) = \{p(t), q(t)\}$ is alternating oscillation and has no common zeros. Or written as

$$z_t = -JH_z, \quad z_0 = (p_0, q_0)^T, \quad J^{-1} = J^T = -J, \quad J^2 = -I, \quad J^T J = I. \tag{8.1}$$

Fengkang(1920-1993), famous Chinese mathematician, proposed(1984) symplectic finite difference (SFD) Z^n at the uniform nodes $t_n = nh, n = 0, 1, 2, 3, ...$ Firstly, found the midpoint formula

$$Z^{n+1} - Z^n = -hJH_z(Z^*), \quad Z^* = (Z^{n+1} + Z^n)/2, \quad Z^0 = (p_0, q_0)^T.$$

Calculating the derivative $X_n = X_n^{n+1} = \frac{dZ^{n+1}}{dZ^n}$ (a square matrix), one has

$$X_n = I - \frac{h}{2} JH_{zz}(Z^*)(I + X_n), \quad so \;\; X_n = (I + C)^{-1}(I - C), \quad C = JA, \; A = \frac{h}{2} H_{zz}.$$

which is a second-order diagonal Pade approximation (i.e. Cayley transform), where H_{zz} and $A = \frac{h}{2} H_{zz}$ are symmetric matrices, h is a small size.

Theorem a. *For an infinitesimal symplectic matrix $C = JA$, then $JC + C^T J = 0$.*

Proof. Directly substitute $C = JA$ into $JC + C^T J = J^2 A + A^T J^T J = -IA + A^T I = -A + A^T = 0$.

Theorem b. *Cayley transform $X = (I + C)^{-1}(I - C)$ is sumplectic matrix, i.e. $X^T JX = J$.*

Proof. The small parameter h ensures that the matrix $(I \pm C)$ is non singular, $C = JA$. Firstly we directly prove an equality

$$(I + C)J(I + C)^T = (I - C)J(I - C)^T, \tag{8.2}$$

Using Theorem a, $JC^T = -CJ$. So the left and right sides of (8.2) become

$$(I+C)(I-C)J = (I-C)(I+C)J.$$

So (8.2) holds. By (8.2) we have $(I-C)^{-1}(I+C)J(I+C)^T(I-C)^{-T} = J$. Take their inverse, as $J^{-1} = -J$, the symplectic conversation $X^TJX = J$ is proved.

The $2m$ order diagonal Pade approximation is used in high-order symplectic schemes, and the above proof is still valid

Now we still return to the midpoint formula and the symplectic observation $X_n^TJX_n = J$. If we take the derivative of the initial value Z^0, using the chain rule of the derivative,

$$\frac{dZ^{n+1}}{dZ^0} \equiv X_0^{n+1} \equiv X_n^{n+1}X_{n-1}^n \cdots X_0^1, \quad and \quad (X_0^{n+1})^T JX_0^{n+1} = J. \tag{8.3}$$

Therefore, in the multiplication structure, X_0^{n+1} is symplectic observation. FengKang and other Chinese scholars have made extensive research on the symplectic scheme and have also extended it to partial differential H-systems. A few years later, San-Serna (1989) found that Runge-Kutta method with m-point Gauss quadrature, is also a symplectic scheme, with highest accuracy $O(h^{2m})$ at nodes, which is one of the best symplectic schemes, and the midpoint formula is its special case.

Why is the symplectic algorism important? Because the symplectic algorism has the smallest deviation in long-time calculations. For non-symplectic algorism, e.g. the trapezoidal formula has the long-time accumulation error $O(t_n^2h^2)$. FengKang (1985) said a humorous sentence, *"Using the trapezoidal formula to calculate the satellite's orbit around the Earth and falling on the ground after dozens of revolutions"*. He had proposed an important conjecture as follows.

Fengkang conjecture(FC). *For $2m$-order symplectic algorism Z^n, their deviation linearly increases with t_n*

$$|Z^n - z(t_n)| \leq Ct_nh^{2m}. \tag{8.4}$$

But FengKang did not publish it when he was alive. We are announcing it here. The symplectic algorithm generally cannot maintain energy conservation(but with error $O(t_nh^{2m})$.

We studied m-degree continuous finite element(CFM) and found that CFM can preserve energy at node t_n and also satisfies FC, but cannot preserve symplecticity(also with error $O(t_nh^{2m})$). Therefore the both is twin brothers.

FC is an error accumulation process (additive structure) that is not directly related to symplectic (multiplicative structure). So far, FC has not been proved using the symplectic algorithm. But we found that by using the property of CFM maintaining energy at node t_n, the corresponding FC can be proved at least in one-dimensional problems. Therefore, there is an interesting comparison here: using $\zeta(s)$ cannot prove RC, while $\xi(z)$ can because it has a product expression. Using the symplectic algorithm cannot prove FC, but using CFM is possible because it preserves energy. This once again confirms FengKang's famous saying: *"Equivalent is not necessarily equi-effect"*.

§8.2 Continuous finite elements of Schrödinger equation

We consider a simplest linear Schrödinger equation with homogenous boundary value

$$u_t = -\Delta v,\ u_t = \Delta u, \quad u(0,x) = u_0(x),\ v(0,x) = v_0(x), \tag{8.5}$$

which also is anti-symmetrical structure. It has two conservation laws. Multiply by u and v separately, then add, integrating on $Q_T = [0,T] \times \Omega$ gets mass conservation

$$\frac{1}{2}\int_\Omega (u^2+v^2)dx|_{t=0}^{t=T} = \int_{Q_T} (u_t u + v v_t)dtdx = \int_{Q_T} (-\Delta v u + \Delta u u)dtdx = 0. \tag{8.6}$$

Multiply by v_t and u_t separately, and subtract. Integrating gets energy conservation

$$0 = \int_{Q_T} (u_t v_t - v_t u_t)dtdx = -\int_{Q_T} (\Delta v v_t + \Delta u u_t)dtdx = \frac{1}{2}\int_\Omega (|\nabla u|^2 + |\nabla v|^2)dx|_{t=0}^{t=T}. \tag{8.7}$$

We (2008) studied the finite element method of Schrödinger equation, $z = \{u, v\}$. Define inner product $(u,v) = \int_\Omega uvdx$. Subdivide time by nodes $t_j = jk$, with step size k. Partition region $\Omega^h = \Omega$, by step size h. Define m-degree time-space continuous finite element $W = \{U, V\}$ satisfying the variational equation

$$\int_0^{t_n} ((U_t,\xi) + (V_t,\eta))dtdx = \int_0^{t_n} ((\nabla V, \nabla\xi) - (\nabla U, \nabla\eta))dtdx, \quad \xi,\zeta \in P_{m-1}(t). \tag{8.8}$$

Since the degrees of freedom must match in each $[t_j, t_{j+1}]$, the test functions ξ, η can only be the polynomials $P_{m-1}(t)$ of degree $m-1$. We can take $\xi = V_t, \zeta = -U_t$ in (8.8), and obtain energy conservation (8.7). But we cannot take $\xi = U, \eta = V$ in (8.8), for getting (8.6). In theoretical analysis (using trace theorem in Sobolev space), the error estimate only is $O(k^{2m-1/2})$, which reduces the order by half. This defect has been solved by us, see §8.3.

§8.3 Local projection of test function

The author (2015) proposed new idea.

Test function projection(TFP). *For any function $W(t)$ in the time element $[t_j, t_{j+1}]$, define its local projection $P^k W \in P_{m-1}$ satisfying orthogonal relation*

$$\int_{t_j}^{t_{j+1}} (W - P^k W)\eta(t)dt = 0, \quad \eta \in P_{m-1} = polynomials\ of\ degree\ m-1. \tag{8.9}$$

If taking $\xi = P^k U, \eta = P^k V$ in (8.8), then the left side is positive definite

$$\int_0^{t_n} ((U_t, P^k U) + (V_t, P^k V))dt = \int_0^{t_n} ((U_t, U) + (V_t, V))dt = \frac{1}{2}(||U||^2 + ||V||^2)|_0^{t_n}. \tag{8.10}$$

and all the antisymmetric derivative terms on the right are canceled out(an exciting property)

$$\int_0^{t_n} ((\nabla V, \nabla P^k U) - (\nabla U, \nabla P^k V))dt = \int_0^{t_n} ((\nabla P^k V, \nabla P^k U) - (\nabla P^k U, \nabla P^k V))dt = 0. \tag{8.11}$$

Therefore we obtained the mass conservation (8.6) at the time-nodes t_n.

I and my colleagues [22, 23] have studied Maxwell equations in meta-materials (e.g. the surface layer material of stealth aircraft), use the above method to obtain a full order estimate $O(k^{2m})$ at the nodes t_n for time continuous finite element (CG). Besides, the discontinuous space-time finite element(DG) has also been studied.

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