Research Article

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Entangled Photon-Electron following Thermo-Field Dynamics Method

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Abstract
Thermo-field dynamics (TFD) is a method linking quantum information to statistical thermodynamics, focus directly on the state because the states in the tilde space in TFD play a role of tracer of the initial states by defining an extended density matrix described on the double Hilbert space. The study of quantum entanglement is more complicated, so the TFD method makes it easier, this is the subject of this paper. We use Thermo-field Dynamics method to study entanglement entropies and transition probability of a Photon-electron system between equilibrium with heat bath, and non- equilibrium, based on the dissipative von Neumann equation. Numerically, at the equilibrium and by controlling the system through temperature, we show that entanglement entropies verify the Nernst's theorem. By increasing the quantization of a single-mode electromagnetic field, entanglement, and transition probability decrease. At non-equilibrium, we show that entanglement entropies obey the second law of thermodynamics, and the emission of a photon decrease the probability of detecting the electron at a quantum level.

Keywords
Photon-Electron System, Thermo-Field Dynamics Method, Entanglement, Transition Probability

Introduction

Entanglement is an exotic device between correlated particles that has no classical significance. Like the development of quantum information science [1], entanglement has acquired new importance in quantum communication: cryptography [2], quantum coding and teleportation [3], quantum computing [4] etc.... One of the most important entangled quantum systems is the photon. Quantum entanglement of photons could pave the way for insights into quantum optics and other areas of physics such as modern quantum technologies: quantum metrology [5], quantum information [6], simulate quantum transport [7], linear optical quantum computing (LOQC) [8-10], etc. Entangled photons are generated through the interaction between an electromagnetic field and matter. The performance of generating entangled photons depends on several factors. The investigation of the interaction between photons and electrons is therefore an important part of modern physics specifying for example research on plasmons, it can be managed through the interaction of electrons and photons [11], it is possible also to use electrons to change the quantum statistics of photons [12], and to transfer the quantum statistics of photons to electrons [13].

Eigenenergy of Entangled Photon-Electron System

The stationary Hamiltonian of the photon-electron system can be read as [14],

$$H = \frac{1}{2M_{eff}} P_{eff}^2 + \frac{1}{2M} P_z^2 + \hbar\Omega \left(A_0^+ A_0 + \frac{1}{2} \right), \tag{2.1}$$

In expression (2.1), P_{eff}, P_z are the transverse components of the momentum of the electron. $M_{eff} = \frac{1+\frac{\omega_p^2}{2\omega^2}}{1-\frac{\omega_p^2}{2\omega^2}}$

is the transverse mass, M is the mass of the electron and $\Omega = \omega\left(1+\frac{\omega_p^2}{2\omega^2}\right)$ is the frequency, where

ω is the frequency of the mode, ω_p is the frequency of the plasma for an electron density 1 and is given as $\omega_p^2 = \frac{4\pi e^2}{ML^3}$ We read L3 is the quantization volume.

Their corresponding eigenenergy is

$$E_{\vec{k},n_0} = \frac{\hbar^2}{2M_{eff}}k_{eff}^2 + \frac{\hbar^2}{2M}k_z^2 + \hbar\Omega\left(n_0+\frac{1}{2}\right), \tag{2.2}$$

k_{eff}, k_z are the wave vector k components correspond to the projections on the y and z axes.

Expression (2.1) can be rewritten in the Fock space, in the base $|n_0; k_i$ by

$$H = \sum_{n_0=0}^{+\infty}\int d^2k_i\left[\frac{\hbar^2}{2M_{eff}}k_{eff}^2 + \frac{\hbar^2}{2M}k_z^2 + \hbar\Omega\left(n_0+\frac{1}{2}\right)\right]|n_0;k_i\rangle\langle n_0;k_i|, \tag{2.3}$$

To examine the statistical properties of the photon-electron system, we compute the corresponding partition function

$$\begin{aligned} Z =& Tr_{1,2}e^{-\beta H} \\ &= \frac{2\pi}{\beta\hbar^2}\sqrt{M_{eff}M}\frac{\exp\left(-\frac{\beta\hbar}{2}\Omega\right)}{(1-\exp(-\beta\hbar\Omega))}, \end{aligned} \tag{2.4}$$

Entanglement and Transition Probability of System at Equilibrium with Heat Bath

To move on the process of entanglement, we define the density matrix as

$$\begin{aligned} \rho_{eq} =& \frac{e^{-\beta H}}{Z} \\ &= \frac{1}{Z}\sum_{n_0=0}^{+\infty}\int d^2k_i \exp\left[-\frac{\beta\hbar^2}{2M_{eff}}k_{eff}^2 - \frac{\beta\hbar^2}{2M}k_z^2 - \beta\hbar\Omega\left(n_0+\frac{1}{2}\right)\right]|n_0;k_i\rangle\langle n_0;k_i|, \end{aligned} \tag{3.1}$$

The wave vector corresponding to the statistical state is given following expression (3.1) by

$$\begin{aligned} |\psi\rangle =& \sum_{n_0=0}^{+\infty}\int d^2k_i\sqrt{\rho_{eq}}|n_0;k_i\rangle\langle n_0;k_i| \\ &= \frac{1}{Z}\sum_{n_0=0}^{+\infty}\int d^2k_i \exp\left[-\frac{\beta\hbar^2}{2M_{eff}}k_{eff}^2 - \frac{\beta\hbar^2}{2M}k_z^2 - \frac{\beta\hbar}{2}\Omega\left(n_0+\frac{1}{2}\right)\right]|n_0;k_i\rangle|\tilde{n}_0;\tilde{k}_i\rangle, \end{aligned} \tag{3.2}$$

The extended density matrix defines by the relation between the statistical state and their conjugate as

$$\rho = |\psi\rangle\langle\psi| \tag{3.3}$$

Consequently, we obtain

$$\begin{aligned} |\rho\rangle =& \frac{1}{Z}\sum_{m_0=0}^{+\infty}\sum_{n_0=0}^{+\infty}\int d^2k_i d^2k_i^{'} \exp\left[-\frac{\beta\hbar^2}{2M_{eff}}\left(k_{eff}^2 + k_{eff}^{'2}\right) - \frac{\beta\hbar^2}{2M}\left(k_z^2 + k_z^{'2}\right) - \frac{\beta\hbar}{2}\Omega\left(m_0+n_0\right)\right] \\ & |m_0;k_i\rangle\langle n_0;k_i^{'}||\tilde{m}_0;\tilde{k}_i\rangle\langle\tilde{n}_0;\tilde{k}_i^{'}|, \end{aligned} \tag{3.4}$$

The reduced density matrix of particle 1 (photon) is obtained by computing the trace of the extended density matrix (3.4) on particle 2 (electron).

$$\begin{aligned} \rho_{0i} =& Tr_{k_i}\rho \\ &= \frac{1}{Z}\sum_{m_0=0}^{+\infty}\sum_{n_0=0}^{+\infty}\left(\frac{4\pi}{\beta\hbar^2}\right)^{\frac{1}{2}}\left(\frac{4\pi}{\beta\hbar^2}\right)^{\frac{1}{2}}\sqrt{M_{eff}M}\exp\left[-\frac{\beta\hbar}{2}\Omega\left(m_0+n_0\right)\right]|m_0\rangle\langle n_0||\tilde{m}_0\rangle\langle\tilde{n}_0| \end{aligned} \tag{3.5}$$

From expression (3.5), we derive the transition probability as

$$P_t = \frac{1}{Z}\left(\frac{4\pi}{\beta\hbar^2}\right)^{\frac{1}{2}}\left(\frac{4\pi}{\beta\hbar^2}\right)^{\frac{1}{2}}\sqrt{M_{eff}M}\exp\left[-\frac{\beta\hbar}{2}\Omega\left(m_0+n_0\right)\right], \tag{3.6}$$

and the von Neumann entropy

$$S_0 = -\,k_B Tr_{0,0}\left(\rho_{0i}\log\rho_{0i}\right)$$
$$= 2\frac{k_B}{\left(\exp\left[-\frac{\beta\hbar}{2}\Omega\right]-1\right)^2}\left(\frac{\beta\hbar\Omega\left(\exp\left[-\frac{\beta\hbar}{2}\Omega\right]+\exp\left[-\beta\hbar\Omega\right]\right)}{\left(\exp\left[-\beta\hbar\Omega\right]-1\right)} - \left(\exp\left[-\beta\hbar\Omega\right]-1\right)\log\left[2\left(\exp\left[-\beta\hbar\Omega\right]-1\right)\right]\right) \tag{3.7}$$

Renyi entropy at order α is defined by the expression

$$S_\alpha = \frac{1}{1-\alpha}\log\left(Tr_i\rho_{0i}^{\alpha}\right),$$

and it is given following expression (3.5) as

$$Tr\rho_i^{\alpha} = \sum_{\sigma_0=0}^{+\infty}\sum_{\tilde{\sigma}_0=0}^{+\infty}\ldots\sum_{\sigma_\alpha=0}^{+\infty}\sum_{\tilde{\sigma}_\alpha=0}^{+\infty}\langle\sigma_0|\langle\tilde{\sigma}_0|\rho_0|\sigma_0\rangle|\tilde{\sigma}_0\rangle\ldots\langle\sigma_\alpha|\langle\tilde{\sigma}_\alpha|\rho_\alpha|\sigma_\alpha\rangle|\tilde{\sigma}_\alpha\rangle$$
$$=\frac{1}{Z^\alpha}\left(\frac{4\pi}{\beta\hbar^2}\right)^{\alpha}\left(M_{eff}M\right)^{\frac{\alpha}{2}}\frac{1}{\left(1-\exp\left(-\frac{\beta\hbar}{2}\Omega\right)\right)^{2\alpha}} \tag{3.8}$$

$$S_\alpha = \frac{1}{1-\alpha}\log\left(2^\alpha\frac{e^{\left(-\alpha\frac{\beta\hbar}{2}\Omega\right)}\left(1+e^{\left(-\frac{\beta\hbar}{2}\right)}\right)^{\alpha}}{\left(1-e^{\left(-\frac{\beta\hbar}{2}\Omega\right)}\right)^{\alpha}}\right) \tag{3.9}$$

To start with the numerical analysis, we set $\hbar = 1$, $\omega = 3.9$, $M = 4.1$, e = 1, kB and we evaluate results of expressions (3.6), (3.7) and (3.9), by considering the parameters β and L.

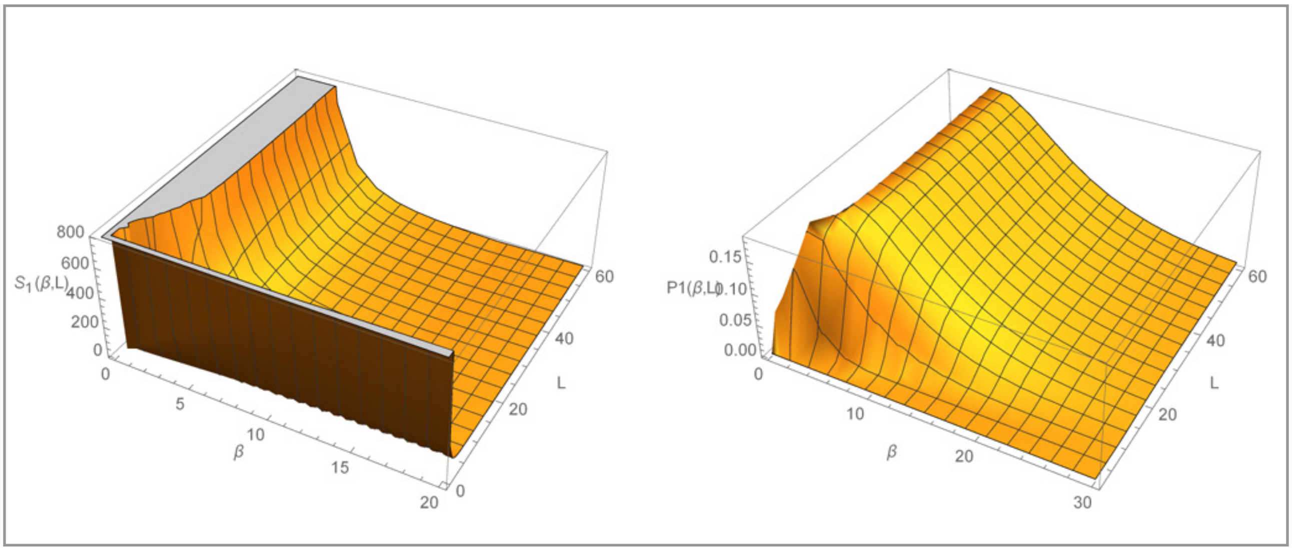

Figure 1: Plots of β − L, (3.7) and (3.6) expresions.

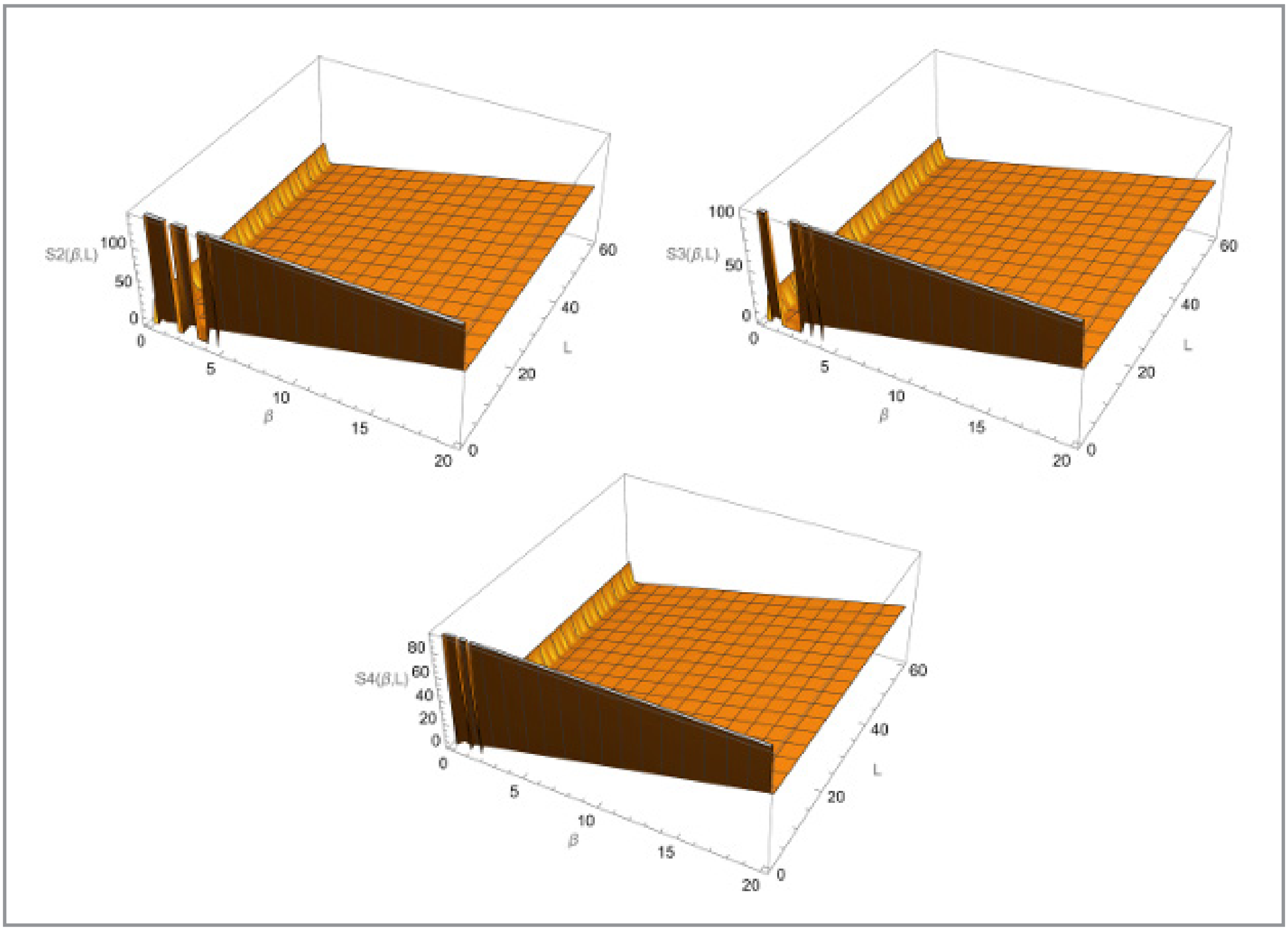

Figure 2: Plots of β-L, (3.9) expression for different values of α, respectively α = 2 S2(β, L), α = 3 S3(β, L), α = 4 S4(β, L).

Fig 1 show that entanglement and transition probability decrease with respect β and L. The creation or destruction of quantum energy at the monomode electromagnetic field will decrease the quantum information and the probability of detecting the electron at the quantum level (4, 4).

Entanglement and Transition Probability of System at Non-Equilibrium with Heat Bath

We can now understand the entangled state of the system at non-equilibrium. The non-equilibrium of the system requires time-dependent control, eventually described by the time-dependent density matrix as

$$\rho_{0neq} = e^{-\zeta t} U^{+}(t) \rho_0 U(t) + \left(1 - e^{-\zeta t}\right) \rho_{0eq}, \tag{4.1}$$

where ρ0 is the density matrix of expression (3.5) at the vacuum state |0, 0>

$$\rho_0 = \frac{1}{Z}\left(\frac{4\pi}{\beta\hbar^2}\right)\sqrt{M_{eff}M}|0,0\rangle\langle 0,0| \tag{4.2}$$

and U (t) is the evolution operator

$$\begin{aligned} U(t) =& e^{\frac{iHt}{\hbar}} \\ =& \sum_{n_0=0}^{+\infty} \int d^2k_i \exp\left[\frac{i}{\hbar}\left[\frac{\hbar^2}{2M_{eff}}k_{eff}^2 + \frac{\hbar^2}{2M}k_z^2 + \hbar\Omega\left(n_0 + \frac{1}{2}\right)\right]t\right] |n_0; k_i\rangle\langle n_0; k_i| \end{aligned} \tag{4.3}$$

consequently we get

$$\rho_{0neq} = \frac{\exp\left[-\frac{\beta\hbar}{2}\Omega\right]}{Z} \sum_{m_0}^{+\infty}\sum_{n_0}^{+\infty} \eta_{m_0,n_0}(t) \exp\left[-\frac{\beta\hbar}{2}\Omega\left(m_0 + n_0\right)\right] |m_0\rangle\langle n_0||\tilde{m}_0\rangle\langle\tilde{n}_0|, \tag{4.4}$$

where

$$\eta_{m_0,n_0}(t) = \left(\left[\sqrt{\left(\left(\delta_{m_0,0}-1\right)e^{-\zeta t}+1\right)}\sqrt{\left(\left(\delta_{n_0,0}-1\right)e^{-\zeta t}+1\right)}\right] + \left(1-e^{-\zeta t}\right)\left(\frac{4\pi}{\beta\hbar^2}\right)\sqrt{M_{eff}M}\right), \tag{4.5}$$

and ρ_{0neq} is the solution of the dissipative von Neumann equation

$$i\hbar\frac{\partial}{\partial t}\rho_{0neq}(t) = \left[H, \rho_{0neq}(t)\right] - \zeta\left(\rho_{0neq}(t) - \rho_{0i}\right). \tag{4.6}$$

Equivalently to (3.6), we obtain from expression (4.4) the transition probability

$$P_t(t) = \frac{\exp\left[-\frac{\beta\hbar}{2}\Omega\right]}{Z}\eta_{m_0,n_0}(t)\exp\left[-\frac{\beta\hbar}{2}\Omega\left(m_0+n_0\right)\right] \tag{4.7}$$

In this case, we write the entanglement von Neumann entropy as

$$\begin{aligned} S_0 =& -k_B Tr_{0,0}\left(\rho_{0neq}\log\rho_{0neq}\right) \\ =& -k_B\frac{\exp\left[-\frac{\beta\hbar}{2}\Omega\right]}{Z}\Bigg(A_1(t)\log\left(\frac{A_1(t)\exp\left[-\frac{\beta\hbar}{2}\Omega\right]}{Z}\right) - \frac{A_2(t)\beta\hbar\Omega\exp\left[\frac{\beta\hbar}{2}\Omega\right]}{2\left(\exp\left[\frac{\beta\hbar}{2}\Omega\right]-1\right)^2} \\ &+ \frac{A_2(t)}{\exp\left[\frac{\beta\hbar}{2}\Omega\right]-1}\log\left(\frac{A_2(t)\exp\left[\frac{\beta\hbar}{2}\Omega\right]}{Z}\right) - \frac{\beta\hbar\Omega\exp\left[\frac{\beta\hbar}{2}\Omega\right]}{2\left(\exp\left[\frac{\beta\hbar}{2}\Omega\right]-1\right)^2}\left(\frac{2A_3(t)}{\exp\left[\frac{\beta\hbar}{2}\Omega\right]-1}+A_2(t)\right) \\ &+ \frac{A_2(t)}{\exp\left[\frac{\beta\hbar}{2}\Omega\right]-1}\log\left(\frac{A_2(t)\exp\left[\frac{-\beta\hbar}{2}\Omega\right]}{Z}\right) + \frac{A_3(t)}{\left(\exp\left[\frac{\beta\hbar}{2}\Omega\right]-1\right)^2}\log\left(\frac{A_3(t)\exp\left[\frac{-\beta\hbar}{2}\Omega\right]}{Z}\right)\Bigg), \end{aligned} \tag{4.8}$$

with

$$A_1(t) = \left(1+\left(1-e^{-\zeta t}\right)\left(\frac{4\pi}{\beta\hbar^2}\right)\sqrt{M_{eff}M}\right),$$

$$A_2(t) = \left(\sqrt{\left(1-e^{-\zeta t}\right)}+\left(1-e^{-\zeta t}\right)\left(\frac{4\pi}{\beta\hbar^2}\right)\sqrt{M_{eff}M}\right),$$

and

$$A_3(t) = \left(\left(1-e^{-\zeta t}\right)+\left(1-e^{-\zeta t}\right)\left(\frac{4\pi}{\beta\hbar^2}\right)\sqrt{M_{eff}M}\right). \tag{4.9}$$

The Renyi entropy is described as

$$S_\alpha = \frac{1}{1-\alpha}\log\left(Tr_i\rho_{neq}^\alpha\right), \tag{4.10}$$

where

$$Tr\rho_{neq}^\alpha = \frac{1}{\left(\frac{2\pi}{\beta\hbar^2}\right)^\alpha\left(M_{eff}M\right)^{\frac{\alpha}{2}}}\frac{1}{\left(1-e^{\left(-\frac{\beta\hbar}{2}\Omega\right)}\right)^{2\alpha}\left(1-e^{\left(-\beta\hbar\Omega\right)}\right)^\alpha}\eta_{\sigma_0,\tilde{\sigma}_0}(t)\ldots\eta_{\sigma_\alpha,\tilde{\sigma}_\alpha}(t) \tag{4.11}$$

Consequently, (4.10) give

$$S_\alpha = \frac{-\alpha}{1-\alpha}\log\left(\left(\frac{2\pi}{\beta\hbar^2}\right)\left(M_{eff}M\right)^{\frac{1}{2}}\right) + \frac{1}{1-\alpha}\log\left(\frac{1}{\left(1-e^{\left(-\frac{\beta\hbar}{2}\Omega\right)}\right)^{2\alpha}\left(1-e^{\left(-\beta\hbar\Omega\right)}\right)^\alpha}\eta_{\sigma_0,\tilde{\sigma}_0}(t)\ldots\eta_{\sigma_\alpha,\tilde{\sigma}_\alpha}(t)\right) \tag{4.12}$$

We set $\hbar = 1$, $\omega = 3.8$, $M = 3.3$, $e = 1$, $L = 3$ and $kB = 1$.

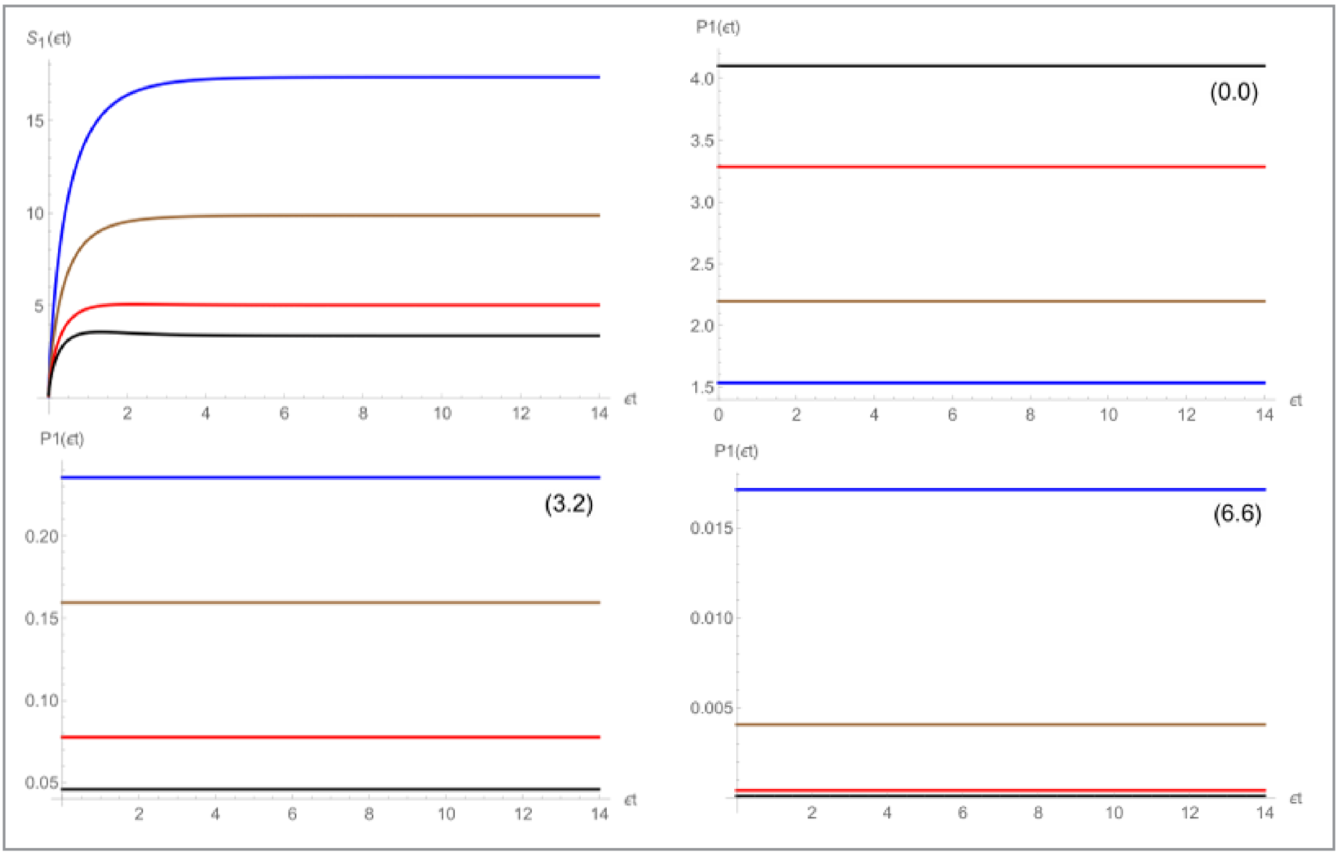

Figure 3: Plot of expressions (4.7) and (4.8) for different values of β respectively β = 0.5 (blue solid line), β = 0.7 (brown solid line), β = 1 (red solid line), β = 1.7 (black solid line) and the couple (m0, n0) for example m0 = 3, n0 = 2 are denoted (3, 2).

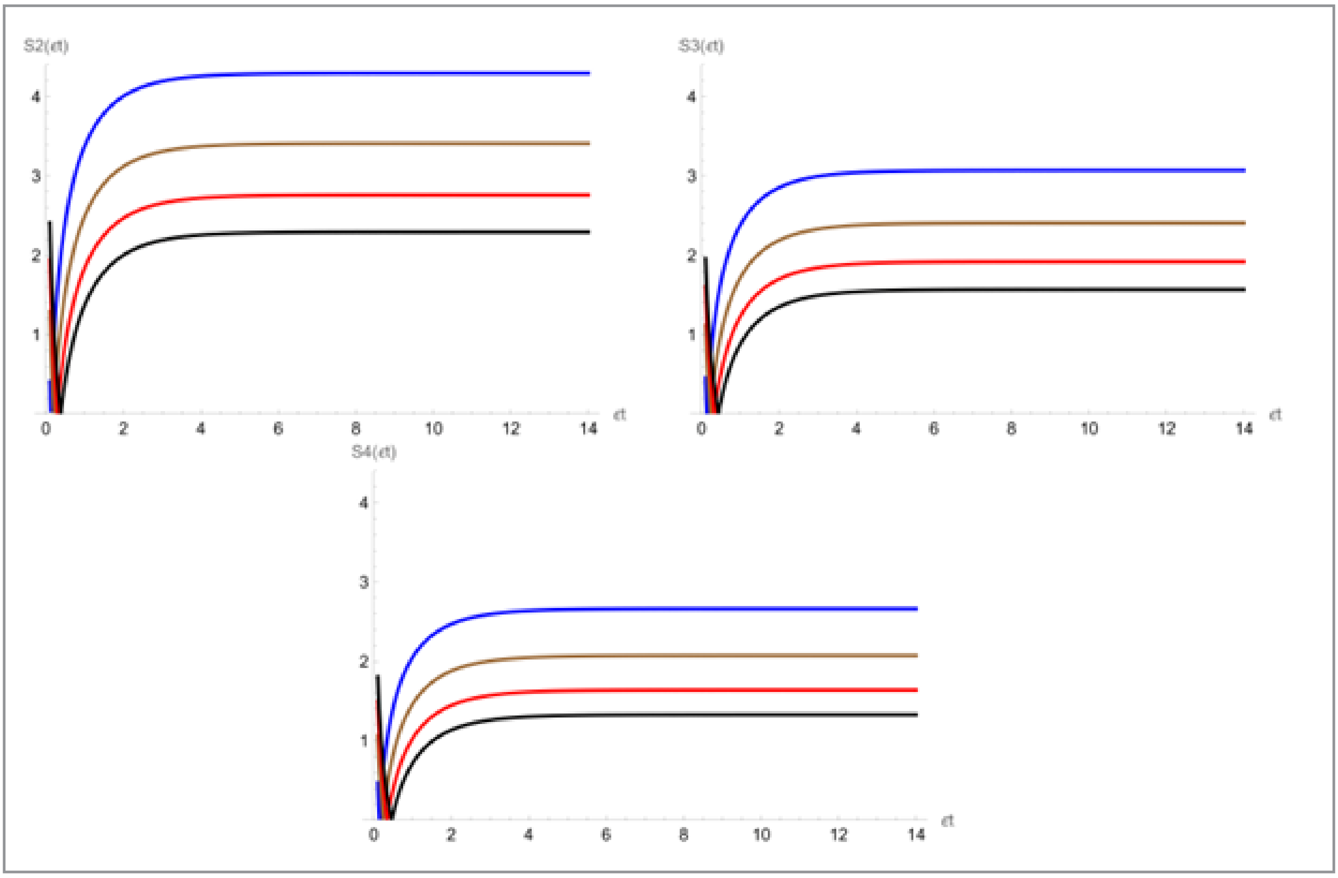

Figure 4: Plot of expressions (4.12) for different values of β respectively β = 0.5 (blue solid line), β = 0.7 (brown solid line), β = 1 (red solid line), β = 1.7 (black solid line) and different values of α, respectively α = 2 S2(ζ t), α = 3 S3(ζ t), α = 4 S4(ζ t).

Clearly, entropy obeys the second law of thermodynamics. The passage of the electron from a higher energy level will decrease the probability of detecting it at that level.

Observe Figs2 and 4, by increasing α, entanglement is reduced, reflecting a less dispersed configuration of the harmonic oscillator while showing that the overall correlations remain large at small α.

To study entanglement of different quantum system, the TFD method is applied in refs [15-17].

Conclusion

This work deal with entanglement entropies and transition probability of photon-electron system used the thermo-field dynamics method at equilibrium and non-equilibrium with heat bath. Results of entangle- ment entropies to be consistent with thermodynamics laws. At equilibrium, the less the electromagnetic field is quantified, the more the system is entangled, the more the probability of detecting the system in a well-specified quantum state increase. At non-equilibrium, excitation of the electron disturbs its transition probability.

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