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The Energy-Momentum Tensor of Electromagnetism Revisited at the Light of Noether

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Abstract
Maxwell's energy-momentum tensor cannot be used as is in Einstein's equation for general relativity in the presence of sources. In this case, its four-divergence is non-zero, whereas the tensors in the equation must have this characteristic. However, the sun emits streams of ionized particles, gases are present within galaxies, material transfers in the form of ionized plasma can occur in binary star systems, and so on. These charges cannot be neglected in the study of space-time in these regions. Here, we present a modified version of this tensor that eliminates this aw, provided that the potentials satisfy the Lorenz gauge. The distribution of sources will also be analyzed in light of fluid mechanics, allowing us to account for its influence in terms of generated pressure.

Keywords
Energy-Momentum, Electromagnetism, Noether's Theorem

Introduction

In the framework of Noether's theorem, an energy-momentum tensor is canonical if the four-divergence is zero. This corresponds to the conservation of energy and momentum. Unfortunately, the energy-momentum tensor of Maxwell commonly used satisfies this condition only in the absence of charges and currents and therefore cannot be used as is in presence of sources.

However, the conventional Maxwell energy-momentum tensor satisfies this conservation law exclusively in source-free regimes, rendering it inadequate for systems containing charges and currents. This limitation becomes critical in general relativity, where a self-consistent description of spacetime geometry at galactic or stellar scales necessitates incorporating electromagnetic interactions between charged matter and fields.

To address this, the proposed formulation explicitly integrates the contributions of charged particle dynamics and their coupling to electromagnetic potentials. The tensor presented in this article takes these effects into account and is thus likely to pave the way for better modeling and, in turn, improve the understanding and knowledge of the physics of these regions. The system is modeled as a perfect fluid comprising charged particles in a comoving frame, with macroscopic quantities derived through statistical averaging over microscopic constituents.

Crucially, the symmetry requirements of the energy-momentum tensor impose the Lorenz gauge condition, ensuring both gauge invariance and compatibility with the geometric structure of Einstein's field equations.

The derivation proceeds via a two-stage methodology:

1. **Special Relativistic Foundation:** Initial construction in Minkowski space-time establishes core tensor properties through explicit calculation of Noether currents under spacetime translations and gauge transformations.
2. **General Relativistic Generalization:** Covariant promotion to curved space- time, preserving the tensor's divergence structure.

This approach resolves key decencies of traditional formulations by:

- Maintaining symmetry without ad hoc improvement terms,
- Including explicit source terms for charge-current distributions,
- Ensuring compatibility with the Bianchi identities in Einstein's equations.

The resulting tensor enables consistent modeling of energy-momentum exchange between electromagnetic fields and charged matter in relativistic astrophysical systems, providing a critical tool for analyzing phenomena such as:

- Plasma dynamics in accretion disks
- Magnetized compact objects
- Large-scale structure formation with cosmic magnetic fields

Further development requires addressing quantum field theoretic aspects of the stress-energy tensor in curved spacetime, particularly renormalization procedures.

Nevertheless, this classical framework establishes essential groundwork for unifying macroscopic gravitational e ects with microscopic electromagnetic interactions in relativistic systems.

Charged Particles Distribution

We will consider that distribution can be likened to an incompressible inviscid relativistic fluid. As a consequence, its rest mass density ρm is constant, and the speed divergence is zero $\boldsymbol{\nabla}.\boldsymbol{v} = 0$ and $\partial_\mu v^\mu = 0$.

This gives us two alternatives for v : either it does not depend explicitly on spatial coordinates, or it derives from the curl of a potential vector. For each kind of particles, the quantities m and q are constants, yielding the ratio $\frac{\rho_e}{\rho_m} = \frac{q}{m}$.

The expression for the canonical relativistic energy-momentum tensor can be found in [1-4] and is

$$T_{fl}{}^{\mu\nu} = (\mathcal{E} + P)\, u^\mu u^\nu - \eta^{\mu\nu} P \tag{1}$$

with the symbols de ned previously. In a comoving rest frame, this tensor is diagonal: diagram (E , P, P, P).

The conservation laws of energy and momentum imply that $\partial_\mu T_{fl}{}^{\mu\nu} = 0$.

The evidence that this relationship is verified is available in [1-4].

Construction of a Canonical Tensor, Neither Symmetric Nor Anti-Symmetric

Let us start with the Lagrangian density of electromagnetism in the presence of charges, augmented by the Lagrangian density of incompressible relativistic fluid Lfl, as we consider it to represent the particle distribution. This provides

$$\mathcal{L}\left(\phi_\alpha, \partial_\mu \phi_\alpha\right) = -\frac{1}{4\mu_0} F_{\mu\alpha} F^{\mu\alpha} - \phi_\alpha j^\alpha + \mathcal{L}_{fl}$$

with $F_{\mu\alpha} = \partial_\mu \phi_\alpha - \partial_\alpha \phi_\mu$.

Given that we already know the canonical tensor of the uid by (1), we will not further develop Lfl; this approach is beyond the scope of this publication. However, for the electromagnetic part, the contravariant canonical tensor is obtained from the relation initial tensor $\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_\alpha\right)} \partial^\nu \phi_\alpha - \eta^{\mu\nu} \mathcal{L}$, which allows us to write an initial tensor.

$$\begin{aligned}\Theta^{\mu\nu}\left(\phi_\alpha, \partial_\mu \phi_\alpha, u_\alpha\right) = &-\frac{1}{\mu_0} F^\mu{}_\alpha \partial^\nu \phi^\alpha + \frac{\eta^{\mu\nu}}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} + \eta^{\mu\nu} \phi_\alpha j^\alpha \\ &+ (\mathcal{E} + P)\, u^\mu u^\nu - \eta^{\mu\nu} P\end{aligned} \tag{2}$$

This tensor will be made symmetric in the next section.

Angular Momentum Conservation

If the canonical tensor $\Theta^{\mu\nu}$ is not symmetric, the angular momentum tensor is written as $M^{\alpha\mu\nu} = x^\mu \Theta^{\alpha\nu} - x^\nu \Theta^{\alpha\mu} + S^{\alpha\mu\nu}$, where $S^{\alpha\mu\nu}$ is an intrinsic spin tensor. By construction, $S^{\alpha\mu\nu}$ is anti-symmetric in (μ, ν).

The angular momentum conservation implies ∂αMαμν = 0, and thus, considering that $\partial_\alpha \Theta^{\alpha\mu} = \partial_\alpha \Theta^{\alpha\mu} = 0$, the relation must be satisfied $\Theta^{\mu\nu} - \Theta^{\nu u} = \partial_\alpha S^{\alpha\nu\mu}$.

The anti-symmetric part of the tensor to be made symmetric must therefore be expressible as the four-divergence of a rank-3 tensor if one wishes to make it symmetric.

The calculation of the anti-symmetric part yields

$$\Theta^{\mu\nu} - \Theta^{\nu\mu} = \frac{1}{\mu_0}\left(\partial^\nu \phi_\alpha \partial^\mu \phi^\alpha - \partial_\alpha \phi^\nu \partial^\mu \phi^\alpha - \partial^\mu \phi_\alpha \partial^\nu \phi^\alpha + \partial_\alpha \phi^\mu \partial^\nu \phi^\alpha\right)$$

Examination of this relationship shows that the only way to express it solely in the form of a four-divergence involves $\partial_\alpha \phi^\alpha = 0$, the Lorenz gauge.

Under this condition, we finally obtain $S^{\alpha\mu\nu} = \frac{1}{\mu_0}\left(\phi^\mu \partial^\nu \phi^\alpha - \phi^\nu \partial^\mu \phi^\alpha\right)$

We will now apply the Belinfante method [5].

Let us define $\theta^{\alpha\mu\nu} = \frac{1}{2}\left(S^{\alpha\nu\mu} + S^{\nu\alpha\mu} - S^{\mu\nu\alpha}\right)$; its computation yields $\theta^{\alpha\mu\nu} =$ $\frac{1}{2\mu_0}\left(\phi^\mu \partial^\nu \phi^\alpha - \phi^\nu \partial^\mu \phi^\alpha + \phi^\mu \partial^\alpha \phi^\nu - \phi^\alpha \partial^\nu \phi^\nu - \phi^\alpha \partial^\nu \phi^\mu + \phi^\nu \partial^\alpha \phi^\mu\right)$.

The symmetrical tensor Tμν is obtained by subtracting

$$\partial_\alpha \theta^{\alpha\mu\nu} = \frac{1}{2\mu_0} \partial_\alpha\left(\phi^\mu \partial^\nu \phi^\alpha + \phi^\nu \partial^\mu \phi^\alpha - \phi^\alpha \partial^\nu \phi^\nu - \phi^\alpha \partial^\nu \phi^\mu\right) - \frac{1}{\mu_0} F^\mu{}_\alpha \partial^\alpha \phi^\nu + \frac{1}{2}\left(\phi^\mu j^\nu + \phi^\nu j^\mu\right)$$

from egation (2), yielding

$$\begin{aligned}T^{\mu\nu}\left(\phi_\alpha, \partial_\mu \phi_\alpha, u_\alpha\right) = &\frac{1}{\mu_0} F^\mu{}_\alpha F^{\alpha\nu} + \frac{\eta^{\mu\nu}}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta} + (\mathcal{E} + P)\, u^\mu u^\nu \\ &-\frac{1}{2} \phi^\mu j^\nu - \frac{1}{2} \phi^\nu j^\mu + \eta^{\mu\nu}\left(\phi_\alpha j^\alpha - P\right) \\ &-\frac{1}{2\mu_0} \partial_\alpha\left(\phi^\mu \partial^\nu \phi^\alpha + \phi^\nu \partial^\mu \phi^\alpha - \phi^\alpha \partial^\nu \phi^\nu - \phi^\alpha \partial^\nu \phi^\mu\right)\end{aligned} \tag{3}$$

Even though the term $\partial_\alpha\left(\phi^\mu \partial^\nu \phi^\alpha + \phi^\nu \partial^\mu \phi^\alpha - \phi^\alpha \partial^\nu \phi^\nu - \phi^\alpha \partial^\nu \phi^\mu\right)$ is a four- divergence term, it cannot be eliminated in any way. If this is the case, $\partial_\mu T^{\mu\nu} = 0$ would no longer be satisfied.

We also see from that the energy-momentum tensor of Maxwell appears spontaneously in the first two terms of the right-hand side [3].

Hamiltonian Density

In our canonical tensor, the Hamiltonian density is provided by the term T 00. Calculating this element leads to the following expression:

$$T^{oo} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \boldsymbol{A}.\boldsymbol{J} + \frac{\mathcal{E} + P\frac{V^2}{c^2}}{1 - \frac{V^2}{c^2}}$$

which corresponds well to the sum of the electromagnetic energy and the fluid energy densities.

To find the nonrelativistic limit (v << c), it is convenient to express the total energy density E as the sum of the mass energy $\rho_m c^2$ and the internal energy $\rho_m \varpi$ densities.

In doing so, it brings us to

$$T^{oo} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) - \boldsymbol{A}.\boldsymbol{J} + \rho_m c^2 + \rho_m \varpi + \frac{1}{2}\rho_m v^2$$

where the contribution of $P\frac{V^2}{c^2}$ is not taken into account as negligible with respect to the other terms.

In a comoving frame, as v = 0, A.J = ρeA.v = 0 and the Hamiltonian density reduces to $T^{00} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) + \mathcal{E}.$.

$T^\mu{}_\mu$ in special relativity

In Minkowski spacetime, $T^\mu{}_\mu$ is readily calculated, which leads to $T^\mu{}_\mu = \mathcal{E} - 3\left(P - \phi_\alpha j^\alpha\right)$

where $\phi_\alpha j^\alpha$, which is the dot product of two 4-vectors, is a relativistic in- variant.

In a comoving frame, as v = 0, $\phi_\alpha j^\alpha$ reduces to $\rho_e \Phi$ and $T^\mu{}_\mu$ reduces to E − 3 ($P - \rho_e \Phi$).

In astrophysical scenarios, relativistic fluids are often encountered in extreme environments, such as close to massive objects like black holes or in high-energy events like supernovae. In these situations, the speeds of particles or matter involved are comparable to the speed of light.

For a relativistic perfect fluid, one can demonstrate [2] that the trace of its energy-momentum tensor equals γ−1ρmc2.

As for ultra-relativistic uids γ−1 ≈ 0 and hence E ≈ 3P . It means that $T^\mu{}_\mu$ will then depends almost only on the electromagnetism through $3\phi_\alpha j^\alpha$ or $3\rho_e \Phi$ in a co-moving rest frame.

For baryonic matter, Tμ must be positive or null which implies $E \geq 3$ (P − $\phi_\alpha j^\alpha$).

Violation of this condition would indicate that we are in the presence of exotic matter.

As it's always possible to find a region of space small enough to use the usual Minkowski spacetime coordinates, it would be interesting to check whether $3\phi_\alpha j^\alpha$ does not become (transiently) negative enough for $T^\mu{}_\mu$ to become negative too, especially in regions where ultra-relativistic fluids are present.

General Relativity

In the presence of charges, the Maxwell stress-energy momentum tensor is not suitable for use, as is the case for Einstein's equation, because its 4-divergence is not null. On the other hand, the tensor that we developed in does not suffer from this defect and can be introduced directly, as is Einstein's equation [3].

Its covariant form, adapted to general relativity, is given by

$$\begin{aligned} T_{\mu\nu}\left(\phi_\alpha, \partial_\mu\phi_\alpha, u_\alpha\right) = \frac{1}{\mu_0}F_{\mu\alpha}F^\alpha{}_\nu + \frac{g_{\mu\nu}}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} + (\mathcal{E}+P)\,u_\mu u_\nu \\ -\frac{1}{2}\phi_\mu j_\nu - \frac{1}{2}\phi_\nu j_\mu + g_{\mu\nu}\left(\phi_\alpha j^\alpha - P\right) \\ -\frac{1}{2\mu_0}\nabla_\alpha\left(\phi_\mu\nabla_\nu\phi^\alpha + \phi_\nu\nabla_\mu\phi^\alpha - \phi^\alpha\nabla_\mu\phi_\nu - \phi^\alpha\nabla_\nu\phi_\mu\right) \end{aligned} \quad (4)$$

that can be split into the electromagnetic part

$$\begin{aligned} T_{EM\,\mu\nu}\left(\phi_\alpha, \partial_\mu\phi_\alpha\right) = \frac{1}{\mu_0}F_{\mu\alpha}F^\alpha{}_\nu + \frac{g_{\mu\nu}}{4\mu_0}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}\phi_\mu j_\nu - \frac{1}{2}\phi_\nu j_\mu + g_{\mu\nu}\phi_\alpha j^\alpha \\ -\frac{1}{2\mu_0}\nabla_\alpha\left(\phi_\mu\nabla_\nu\phi^\alpha + \phi_\nu\nabla_\mu\phi^\alpha - \phi^\alpha\nabla_\mu\phi_\nu - \phi^\alpha\nabla_\nu\phi_\mu\right) \end{aligned} \quad (5)$$

and the fluid part $T_{fl\,\mu\nu}\left(u_\alpha\right) = (\mathcal{E}+P)\,u_\mu u_\nu - g_{\mu\nu}P$.

Conclusion

Starting from the "noncanonical" Maxwell stress-energy-momentum tensor in the presence of sources, we have augmented it to make it canonical, meaning that its four-divergence is zero in the presence of charged or uncharged massive particles.

The most outstanding result is given by which provides a new SEM tensor for electromagnetism that is suitable for direct use in Einstein's equation, and which we have completed by modeling charges limited to the form of an incompressible, inviscid fluid to give. We also open some paths to detect exotic matter in vicinity of black holes and supernovae. Clearly, this modeling is only a basic example, and further modeling can be envisaged by adding additional tensors, provided that they agree with Einstein's equation of general relativity [4, 5].

By treating these particles as incompressible fluids, we observe that these sources modify the fluid pressure, as can be calculated, for instance, from the ideal gas law.

We hope that this article will contribute to better modeling ionized gas behavior within the interstellar medium and, linked to metrics such as the Schwarzschild or Friedmann Lematre Robertson Walker metrics, improve cosmology understanding.
The price to pay for achieving tensor symmetry is the adherence to the Lorenz gauge. Therefore, it is tempting to consider the Lorenz gauge as a fifth equation of electromagnetism that would complement the four Maxwell's equations.

Data Availability Statement

No Data associated in the manuscript.

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Conflicts of Interests/Competing Interests

I have no conflicts of interests or competing interests to declare.

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